To differentiate such expressions we use the **product rule**, which can be written as:

| Function | Derivative |
|---|---|
| If $y = u \times v$ then | $\frac{dy}{dx} = \frac{du}{dx} \times v + u \times \frac{dv}{dx}$ |
| If $y = f(x) \times g(x)$ then | $\frac{dy}{dx} = f'(x) \times g(x) + f(x) \times g'(x)$ |

**EXAMPLE 19.16**

Differentiate the following

(a) $x^2 \sin(x)$ (b) $(x^3 - 2x + 1)e^x$ (c) $\frac{1}{x}\log_e(x)$

**Solution**

(a) Let $y = x^2 \sin(x)$ so that $u = x^2$ and $v = \sin(x)$. So that $\frac{du}{dx} = 2x$ and $\frac{dv}{dx} = \cos(x)$.

Using the product rule we have $\frac{dy}{dx} = \frac{du}{dx} \times v + u \times \frac{dv}{dx}$

$$= 2x \times \sin(x) + x^2 \times \cos(x)$$

$$= 2x\sin(x) + x^2\cos(x)$$

A useful method to find the derivative of a product makes use of the following table:

| Function | Derivative |
|---|---|
| $u = x^2$ | $\frac{du}{dx} = 2x$ |
| $v = \sin(x)$ | $\frac{dv}{dx} = \cos(x)$ |

$2x\sin(x)$

$x^2\cos(x)$

Adding: $2x\sin(x) + x^2\cos(x)$

(b) Let $y = (x^3 - 2x + 1)e^x$ so that $u = (x^3 - 2x + 1)$ and $v = e^x$.

Then, $\frac{du}{dx} = 3x^2 - 2$ and $\frac{dv}{dx} = e^x$.

$$\frac{dy}{dx} = \frac{du}{dx} \times v + u \times \frac{dv}{dx}$$

Using the product rule: $= (3x^2 - 2) \times e^x + (x^3 - 2x + 1) \times e^x$

$$= (3x^2 - 2 + x^3 - 2x + 1)e^x$$

$$= (x^3 + 3x^2 - 2x - 1)e^x$$

(c) Let $y = \frac{1}{x}\log_e x$ with $u = \frac{1}{x}$ and $v = \log_e x$. This time set up a table:

| Function | Derivative |
|---|---|
| $u = \frac{1}{x}$ | $\frac{du}{dx} = -\frac{1}{x^2}$ |
| $v = \log_e x$ | $\frac{dv}{dx} = \frac{1}{x}$ |

$-\frac{1}{x^2} \times \log_e x$

$\frac{1}{x} \times \frac{1}{x}$

Adding: $\frac{dy}{dx} = -\frac{1}{x^2} \times \log_e x + \frac{1}{x} \times \frac{1}{x}$

$$= -\frac{1}{x^2} \times \log_e x + \frac{1}{x^2}$$

$$= \frac{1}{x^2}(1 - \log_e x)$$

## 19.3.5 DERIVATIVE OF A QUOTIENT OF FUNCTIONS

In the same way as we have a rule for the product of functions, we also have a rule for the quotient of functions. For example, the function

$$y = \frac{x^2}{x^3 + x - 1}$$

is made up of two simpler functions of $x$. Expressions like this take on the general form

$$y = \frac{u}{v} \quad \text{or} \quad y = \frac{f(x)}{g(x)}.$$

For the example shown above, we have that $u = x^2$ and $v = x^3 + x - 1$.

As for the product rule, we state the result.

To differentiate such expressions we use the **quotient rule**, which can be written as:

| Function | Derivative |
|---|---|
| If $y = \frac{u}{v}$ then | $\frac{dy}{dx} = \frac{\frac{du}{dx} \times v - u \times \frac{dv}{dx}}{v^2}$ |
| If $y = \frac{f(x)}{g(x)}$ then | $\frac{dy}{dx} = \frac{f'(x) \times g(x) - f(x) \times g'(x)}{[g(x)]^2}$ |

**EXAMPLE 19.17**

Differentiate the following

(a) $\frac{x^2 + 1}{\sin(x)}$ (b) $\frac{e^x + x}{x + 1}$ (c) $\frac{\sin(x)}{1 - \cos(x)}$

**Solution**

(a) We express $\frac{x^2 + 1}{\sin(x)}$ in the form $y = \frac{u}{v}$, so that $u = x^2 + 1$ and $v = \sin(x)$.

Giving the following derivatives, $\frac{du}{dx} = 2x$ and $\frac{dv}{dx} = \cos(x)$.

Using the quotient rule we have,

$$\frac{dy}{dx} = \frac{\frac{du}{dx} \times v - u \times \frac{dv}{dx}}{v^2}$$

$$= \frac{2x \times \sin(x) - (x^2 + 1) \times \cos(x)}{[\sin(x)]^2}$$

$$= \frac{2x\sin(x) - (x^2 + 1)\cos(x)}{\sin^2(x)}$$

(b) First express $\frac{e^x + x}{x + 1}$ in the form $y = \frac{u}{v}$, so that $u = e^x + x$ and $v = x + 1$ and

$\frac{du}{dx} = e^x + 1$ and $\frac{dv}{dx} = 1$. Using the quotient rule, we have

$$\frac{dy}{dx} = \frac{\frac{du}{dx} \times v - u \times \frac{dv}{dx}}{v^2} = \frac{(e^x + 1) \times (x + 1) - (e^x + x) \times 1}{(x + 1)^2}$$

$$= \frac{xe^x + e^x + x + 1 - e^x - x}{(x+1)^2}$$

$$= \frac{xe^x + 1}{(x+1)^2}$$

(c) Express the quotient $\frac{\sin(x)}{1-\cos(x)}$ in the form $y = \frac{u}{v}$, so that $u = \sin(x)$ and $v = 1 - \cos(x)$. Then $\frac{du}{dx} = \cos(x)$ and $\frac{dv}{dx} = \sin(x)$.

Using the quotient rule, we have

$$\frac{dy}{dx} = \frac{\frac{du}{dx} \times v - u \times \frac{dv}{dx}}{v^2} = \frac{\cos(x) \times (1-\cos(x)) - \sin(x) \times \sin(x)}{(1-\cos(x))^2}$$

$$= \frac{\cos(x) - \cos^2(x) - \sin^2(x)}{(1-\cos(x))^2}$$

$$= \frac{\cos(x) - (\cos^2(x) + \sin^2(x))}{(1-\cos(x))^2}$$

$$= \frac{\cos(x) - 1}{(1-\cos(x))^2}$$

$$= -\frac{(1-\cos(x))}{(1-\cos(x))^2}$$

$$= -\frac{1}{(1-\cos(x))}$$

## 19.3.6 THE CHAIN RULE

To find the derivative of $x^3 + 1$ we let $y = x^3 + 1$ so that $\frac{dy}{dx} = 3x^2$.

Next consider the derivative of the function $y = (x^3+1)^2$.

We first expand the brackets, $y = x^6 + 2x^3 + 1$, and obtain $\frac{dy}{dx} = 6x^5 + 6x^2$.

This expression can be simplified (i.e., factorised), giving $\frac{dy}{dx} = 6x^2(x^3+1)$.

In fact, it isn't too great a task to differentiate the function $y = (x^3+1)^3$.

As before, we expand; $y = x^9 + 3x^6 + 3x^3 + 1$ so that $\frac{dy}{dx} = 9x^8 + 18x^5 + 9x^2$.

Factorising this expression we now have $\frac{dy}{dx} = 9x^2(x^6 + 2x^3 + 1) = 9x^2(x^3+1)^2$.

But what happens if we need to differentiate the expression $y = (x^3+1)^8$ ? Of course, we could expand and obtain a polynomial with 9 terms (!), which we then proceed to differentiate and obtain a polynomial with 8 terms. . . and of course, we can then easily factorise that polynomial (not!). The question then arises, "Is there an easier way to do this?"

We can obtain some idea of how to do this by summarising the results found so far:

| Function | Derivative | (Factored form) |
|---|---|---|
| $y = x^3 + 1$ | $\frac{dy}{dx} = 3x^2$ | $3x^2$ |
| $y = (x^3 + 1)^2$ | $\frac{dy}{dx} = 6x^5 + 6x^2$ | $2 \times 3x^2(x^3 + 1)$ |
| $y = (x^3 + 1)^3$ | $\frac{dy}{dx} = 9x^8 + 18x^5 + 9x^2$ | $3 \times 3x^2(x^3 + 1)^2$ |
| $y = (x^3 + 1)^4$ | $\frac{dy}{dx} = 12x^{11} + 36x^8 + 36x^5 + 12x^2$ | $4 \times 3x^2(x^3 + 1)^3$ |

The pattern that is emerging is that if $y = (x^3 + 1)^n$ then $\frac{dy}{dx} = n \times 3x^2(x^3 + 1)^{n-1}$ .

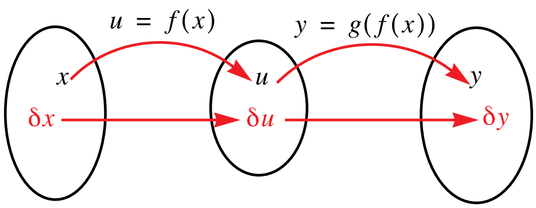
In fact, if we consider the term inside the brackets as one function, so that the expression is actually a composition of two functions, namely that of $x^3 + 1$ and the power function we can write $u = x^3 + 1$ and $y = u^n$ .

So that $\frac{dy}{du} = nu^{n-1} = n(x^3 + 1)^{n-1}$ and $\frac{du}{dx} = 3x^2$ , giving the result $\frac{dy}{dx} = \frac{dy}{du} \times \frac{du}{dx}$ .

Is this a 'one–off' result, or can we determine a general result that will always work?
To explore this we use a graphical approach to see why it might be possible to obtain a general result.

We start by using the above example and then move onto a more general case. For the function $y = (x^3 + 1)^2$ , we let $u = x^3 + 1$ $(= g(x))$ and so $y = u^2$ $(= f(g(x)))$ . We need to find what effect a small change in $x$ will have on the function $y$ (via $u$). i.e., what effect will $\delta x$ have on $y$ ?

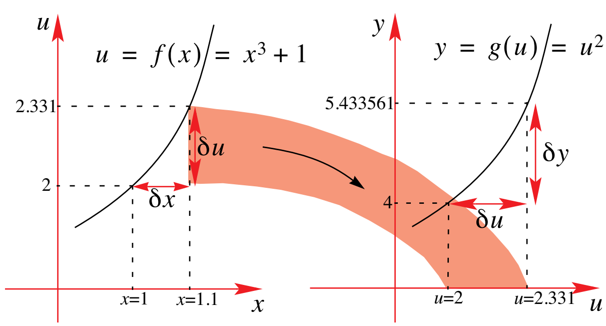
We have a sort of **chain reaction**, that is, a small change in $x$, $\delta x$ , will produce a change in $u$, $\delta u$ , which in turn will produce a change in $y$, $\delta y$ ! It is the path from $\delta x$ to $\delta y$ that we are interested in.




This can be seen when we produce a graphical representation of the discussion so far.

*We start by looking at the effect that a change in x has on u:*

*Then we observe the effect that the change in u has on y :*




*We then have* $\delta x = 1.1 - 1 = 0.1$
*and* $\delta u = 2.331 - 2 = 0.331$ .

*Similarly,* $\delta u = 2.331 - 2 = 0.331$
*&* $\delta y = 5.433561 - 4 = 1.433561$

*Based on these results, the following relationship can be seen to hold:*

$$\frac{\delta y}{\delta x} = \frac{\delta y}{\delta u} \times \frac{\delta u}{\delta x}$$

The basic outline in proving this result is shown in the following argument:

Let $\delta x$ be a small increment in the variable $x$ and let $\delta u$ be the corresponding increment in the variable $u$. This change in $u$ will in turn produce a corresponding change $\delta y$ in $y$.
As $\delta x$ tends to zero, so does $\delta u$. We will assume that $\delta u \neq 0$ when $\delta x \neq 0$. Hence we have that

$$\frac{\delta y}{\delta x} = \frac{\delta y}{\delta u} \cdot \frac{\delta u}{\delta x} \Rightarrow \lim_{\delta x \to 0} \frac{\delta y}{\delta x} = \lim_{\delta x \to 0} \frac{\delta y}{\delta u} \cdot \frac{\delta u}{\delta x}$$

$$= \left(\lim_{\delta x \to 0} \frac{\delta y}{\delta u}\right) \cdot \left(\lim_{\delta x \to 0} \frac{\delta u}{\delta x}\right)$$

$$= \left(\lim_{\delta u \to 0} \frac{\delta y}{\delta u}\right) \cdot \left(\lim_{\delta x \to 0} \frac{\delta u}{\delta x}\right)$$

*Given that:* $\delta x \to 0 \Rightarrow \delta u \to 0$

$$\therefore \frac{dy}{dx} = \frac{dy}{du} \cdot \frac{du}{dx}$$

We then have the result:

$$\frac{dy}{dx} = \frac{dy}{du} \cdot \frac{du}{dx}$$

## Chain rule (composite function notation)

An alternative notation when using the chain rule occurs when the function is expressed in the form of a composite function, i.e., in the form $f \circ g$.

So, if $F = f \circ g$, then $F(x) = f(g(x))$ and $F'(x) = f'(g(x)) \cdot g'(x)$.

That is, the derivative of the composite function $f \circ g$ is $(f \circ g)' = (f' \circ g)g'$, or

$$\frac{d}{dx}(f \circ g) = \frac{df}{du}\frac{du}{dx}, \text{ where } u = g(x).$$

In short, the chain rule provides a process whereby we can differentiate expressions that involve composite functions. For example, the function $y = \sin(x^2)$ is a composition of the *sine* function $\sin(\ )$ and the squared function, $x^2$. So that we would let $u$ (or $g(x)$ ) equal $x^2$, giving $y = \sin(u)$, where $u = x^2$.

The key to differentiating such expressions is to recognise that the chain rule must be used, and to choose the appropriate function $u$ (or $g(x)$ ).

## Using the chain rule

We will work our way through an example, showing the critical steps involved when using the chain rule.

This is highlighted by finding the derivative of the function $y = \sin(x^2)$.

| **Step 1** | **Recognition** |
|---|---|
| This is the most important step when deciding if using the chain rule is appropriate. In this case we *recognise* that the function $y = \sin(x^2)$ is a composite of the sine and the squared functions. | |

| **Step 2** | **Define $u$ (or $g(x)$)** |
|---|---|

Let the 'inside' function be $u$. In this case, we have that $u = x^2$

| **Step 3** | **Differentiate $u$ (with respect to $x$)** |
|---|---|

$$\frac{du}{dx} = 2x$$

| **Step 4** | **Express $y$ in terms of $u$** |
|---|---|

$$y = \sin(u)$$

| **Step 5** | **Differentiate $y$ (with respect to $u$)** |
|---|---|

$$\frac{dy}{du} = \cos(u)$$

| **Step 6** | **Use the chain rule** |
|---|---|

$$\frac{dy}{dx} = \frac{dy}{du}\cdot\frac{du}{dx} = \cos(u)\times 2x = 2x\cos(x^2)$$

**EXAMPLE 19.18**

Differentiate the following functions (a) $y = \log_e(x + \cos x)$

(b) $f(x) = (1 - 3x^2)^4$

**Solution**

(a) Begin by letting $u = x + \cos(x) \Rightarrow \frac{du}{dx} = 1 - \sin(x)$.

Express $y$ in terms of $u$, that is, $y = \log_e u \Rightarrow \frac{dy}{du} = \frac{1}{u}\left(= \frac{1}{x + \cos(x)}\right)$.

Using the chain rule we have $\frac{dy}{dx} = \frac{dy}{du}\cdot\frac{du}{dx} = \frac{1}{x + \cos(x)}\cdot(1 - \sin(x))$

$$= \frac{1 - \sin(x)}{x + \cos(x)}$$

(b) This time we let $g(x) = 1 - 3x^2$, so that $g'(x) = -6x$.

Now let $f(x) = (h\circ g)(x)$ so that $h(g(x)) = (g(x))^4$ and $h'(g(x)) = 4(g(x))^3$.

Therefore, using the chain rule we have

$$\begin{aligned} f'(x) &= (h\circ g)'(x) = h'(g(x))\cdot g'(x) \\ &= 4(g(x))^3\times(-6x) \\ &= -24x(1 - 3x^2)^3 \end{aligned}$$

## Some standard derivatives

Often we wish to differentiate expressions of the form $y = \sin(2x)$ or $y = e^{5x}$ or other such functions, where the $x$ term only differs by a constant factor from that of the basic function. That is, the only difference between $y = \sin(2x)$ and $y = \sin(x)$ is the factor '2'. We can use the chain rule to differentiate such expressions:

Let $u = 2x$, giving $y = \sin(u)$ and so $\frac{dy}{dx} = \frac{dy}{du} \cdot \frac{du}{dx} = \cos(u) \times 2 = 2\cos(2x)$

Similarly,

Let $u = 5x$, giving $y = e^u$ and so $\frac{dy}{dx} = \frac{dy}{du} \cdot \frac{du}{dx} = e^u \times 5 = 5e^{5x}$.

Because of the nature of such derivatives, functions such as these form part of a set of functions that can be considered as having derivatives that are often referred to as standard derivatives. Although we could make use of the chain rule to differentiate these functions, they should be viewed as standard derivatives.

These standard derivatives are shown in the table below (where $k$ is some real constant):

| $y$ | $\frac{dy}{dx}$ |
|---|---|
| $\sin(kx)$ | $k\cos(kx)$ |
| $\cos(kx)$ | $-k\sin(kx)$ |
| $\tan(kx)$ | $k\sec^2(kx)$ |
| $e^{kx}$ | $ke^{kx}$ |
| $\log_e(kx)$ | $\frac{1}{x}$ |

Notice, the only derivative that does not involve the constant $k$ is that of the logarithmic function.

This is because letting $u = kx$, we have $y = \log(u)$ so $\frac{dy}{dx} = \frac{dy}{du} \cdot \frac{du}{dx} = \frac{1}{u} \times k = \frac{1}{\not{k}x} \times \not{k} = \frac{1}{x}$.

## When should the chain rule be used?

A good rule of thumb:

A good **first** rule to follow is: If the expression is made up of a pair of brackets and a power, then, the chances are that you will need to use the chain rule.

As a start, the expressions in the table that follows would require the use of the chain rule. Notice then that in each case the expression can be (or already is) written in 'power form'. That is, of the form $y = [f(x)]^n$.

| Expression | Express in power form | Decide on what *u* and *y* are |
|---|---|---|
| (a) $y = (2x+6)^5$ | Already in power form. | Let $u = 2x+6$ and $y = u^5$ |
| (b) $y = \sqrt{(2x^3+1)}$ | $y = (2x^3+1)^{\frac{1}{2}}$ | Let $u = 2x^3+1$ and $y = u^{\frac{1}{2}}$ |
| (c) $y = \frac{3}{(x-1)^2}, x \neq 1$ | $y = 3(x-1)^{-2}, x \neq 1$ | Let $u = x-1$ and $y = 3u^{-2}$ |
| (d) $f(x) = \sin^2 x$ | $f(x) = (\sin x)^2$ | Let $u = \sin x$ and $f(u) = u^2$ |
| (e) $y = \frac{1}{\sqrt[3]{e^{-x}+e^x}}$ | $y = (e^{-x}+e^x)^{-\frac{1}{3}}$ | Let $u = e^{-x}+e^x$ and $y = u^{-\frac{1}{3}}$ |

## However, this isn't always the case!

Although the above approach is very useful, often you have to recognise when the function of a function rule is more appropriate. By placing brackets in the appropriate places, we can recognise this feature more readily. The examples below illustrate this:

| Expression | Express it with brackets | Decide on what *u* and *y* are |
|---|---|---|
| (a) $y = e^{x^2+1}$ | $y = e^{(x^2+1)}$. | Let $u = x^2+1$ and $y = e^u$ |
| (b) $y = e^{\sin 2x}$ | $y = e^{(\sin 2x)}$ | Let $u = \sin 2x$ and $y = e^u$ |
| (c) $y = \sin(x^2-4)$ | Already in bracket form. | Let $u = x^2-4$ and $y = \sin(u)$ |
| (d) $f(x) = \log_e(\sin x)$ | Already in bracket form. | Let $u = \sin x$ and $f(u) = \log_e(u)$ |

Completing the process for each of the above functions we have:

(a) $\frac{dy}{dx} = \frac{dy}{du}\frac{du}{dx} = e^u \times 2x = 2xe^{x^2+1}$.

(b) $\frac{dy}{dx} = \frac{dy}{du}\frac{du}{dx} = e^u \times 2\cos(2x) = 2\cos(2x)e^{\sin(2x)}$.

(c) $\frac{dy}{dx} = \frac{dy}{du}\frac{du}{dx} = \cos(u) \times 2x = 2x\cos(x^2-4)$.

(d) $\frac{dy}{dx} = \frac{dy}{du}\frac{du}{dx} = \frac{1}{u} \times \cos x = \frac{\cos x}{\sin x} = \cot x$.

We now look at some of the more demanding derivatives, i.e., derivatives which combine at least two rules of differentiation, for example, the need to use both the quotient rule and the chain rule, or the product rule and the chain rule.

**EXAMPLE 19.19**

Differentiate the following

(a) $y = \sqrt{1+\sin^2 x}$ (b) $y = e^{x^3}\sin(1-2x)$ (c) $x \mapsto \frac{x}{\sqrt{x^2+1}}$

**Solution**

(a) Let $y = \sqrt{(1+\sin^2 x)} = (1+\sin^2 x)^{1/2}$. Using the chain rule we have

$$\frac{dy}{dx} = \frac{1}{2}\times\frac{d}{dx}(1+\sin^2 x)\times(1+\sin^2 x)^{-1/2} = \frac{1}{2}\times(2\sin x\cos x)\times\frac{1}{\sqrt{(1+\sin^2 x)}}$$

$$= \frac{\sin x\cos x}{\sqrt{(1+\sin^2 x)}}$$

(b) Let $y = e^{x^3}\sin(1-2x)$. Using the product rule first, we have

$$\frac{dy}{dx} = \frac{d}{dx}(e^{x^3})\times\sin(1-2x)+e^{x^3}\times\frac{d}{dx}(\sin(1-2x))$$

$$= 3x^2e^{x^3}\sin(1-2x)+e^{x^3}\times -2\cos(1-2x)$$

$$= e^{x^3}(3x^2\sin(1-2x)-2\cos(1-2x))$$

(c) Let $f(x) = \frac{x}{\sqrt{x^2+1}} \Rightarrow f'(x) = \frac{\frac{d}{dx}(x)\times\sqrt{x^2+1}-x\times\frac{d}{dx}(\sqrt{x^2+1})}{(\sqrt{x^2+1})^2}$ (*Quotient rule*).

$$= \frac{1\times\sqrt{x^2+1}-x\times\frac{1}{2}\times 2x\times(x^2+1)^{-\frac{1}{2}}}{x^2+1}$$

$$= \frac{\sqrt{x^2+1}-\frac{x^2}{\sqrt{x^2+1}}}{(x^2+1)}$$

$$= \frac{\frac{(\sqrt{x^2+1})^2-x^2}{\sqrt{x^2+1}}}{(x^2+1)}$$

$$= \frac{1}{(x^2+1)\sqrt{x^2+1}}$$

---

## EXAMPLE 19.20

Differentiate the following

(a) $y = \ln\left(\frac{x}{x+1}\right), x>0$ (b) $y = \sin(\ln t)$ (c) $y = x\ln(x^2)$

**Solution**

(a) $y = \ln\left(\frac{x}{x+1}\right) = \ln(x)-\ln(x+1) \therefore \frac{dy}{dx} = \frac{1}{x}-\frac{1}{x+1} = \frac{(x+1)-x}{x(x+1)} = \frac{1}{x(x+1)}$.

Notice that using the log laws to first simplify this expression made the differentiation process much easier.

The other approach, i.e., letting $u = \frac{x}{x+1}$, $y = \ln(u)$ and then using the chain rule would have meant more work – as not only would we need to use the chain rule but also the quotient rule to determine $\frac{du}{dx}$.

(b) Let $u = \ln t$ so that $y = \sin u$.

Using the chain rule we have $\frac{dy}{dt} = \frac{dy}{du}\cdot\frac{du}{dt} = \cos(u)\times\frac{1}{t} = \frac{\cos(\ln t)}{t}$

(c) Here we have a product $x \times \ln(x^2)$, so that the product rule needs to be used and then we need the chain rule to differentiate $\ln(x^2)$.

Notice that in this case we cannot simply rewrite $\ln(x^2)$ as $2\ln(x)$. Why?
Because the functions $\ln(x^2)$ and $2\ln(x)$ might have different domains. That is, the domain of $\ln(x^2)$ is all real values excluding zero (assuming an implied domain) whereas the domain of $2\ln(x)$ is only the positive real numbers. However, if it had been specified that $x > 0$, then we could have 'converted' $\ln(x^2)$ to $2\ln(x)$.

So, $\frac{dy}{dx} = \frac{d}{dx}(x) \times \ln(x^2) + x \times \frac{d}{dx}(\ln(x^2)) = 1 \times \ln(x^2) + x \times \frac{2x}{x^2} = \ln(x^2) + 2$

## A short cut (?)

Once you have practiced the use of these rules and are confident in applying them, you can make use of the following table to speed up the use of the chain rule. Assuming that the function $f(x)$ is differentiable then we have:

| $y$ | $\frac{dy}{dx}$ |
|---|---|
| $\sin[f(x)]$ | $f'(x)\cos[f(x)]$ |
| $\cos[f(x)]$ | $-f'(x)\sin[f(x)]$ |
| $\tan[f(x)]$ | $f'(x)\sec^2[f(x)]$ |
| $e^{f(x)}$ | $f'(x)e^{f(x)}$ |
| $\log_e[f(x)]$ | $\frac{f'(x)}{f(x)}$ |
| $[f(x)]^n$ | $nf'(x)[f(x)]^{n-1}$ |

## 19.3.7 DERIVATIVE OF RECIPROCAL CIRCULAR FUNCTIONS

Dealing with the functions $\sec(x)$, $\cot(x)$ and $\operatorname{cosec}(x)$ is a straight foward matter – simply rewrite them as their reciprocal counterparts. That is, $\sec(x) = \frac{1}{\cos(x)}$, $\cot(x) = \frac{1}{\tan(x)}$ and $\operatorname{cosec}(x) = \frac{1}{\sin(x)}$. Once this is done, make use of the chain rule.

For example, $\frac{d}{dx}(\operatorname{cosec}x) = \frac{d}{dx}\left(\frac{1}{\sin x}\right) = \frac{d}{dx}[(\sin x)^{-1}] = -1 \times \cos x \times (\sin x)^{-2} = -\frac{\cos x}{(\sin x)^2}$.

We could leave the answer as is or simplify it as follows; $-\frac{\cos x}{\sin x \sin x} = -\cot x \operatorname{cosec} x$.

So, rather than providing a table of 'standard results' for the derivative of the reciprocal circular trigonometric functions, we consider them as special cases of the circular trigonometric functions.

**EXAMPLE 19.21**

Differentiate the following

(a) $f(x) = \cot 2x, x > 0$ (b) $y = \sec^2 x$ (c) $y = \dfrac{\ln(\mathrm{cosec}\, x)}{x}$

**Solution**

(a) $f(x) = \cot 2x = \dfrac{1}{\tan 2x} = (\tan 2x)^{-1} \therefore f'(x) = -1 \times 2\sec^2 2x \times (\tan 2x)^{-2}$

$$= -\frac{2\sec^2 2x}{\tan^2 2x}$$

Now, $\dfrac{2\sec^2 2x}{\tan^2 2x} = 2 \times \dfrac{1}{\cos^2 2x} \times \dfrac{1}{\tan^2 2x} = 2 \times \dfrac{1}{\cos^2 2x} \times \dfrac{\cos^2 2x}{\sin^2 2x} = 2\mathrm{cosec}^2 2x$.

And so, $f'(x) = -2\mathrm{cosec}^2 2x$.

(b) $y = \sec^2 x = \dfrac{1}{(\cos x)^2} = (\cos x)^{-2} \therefore \dfrac{dy}{dx} = -2 \times -\sin x \times (\cos x)^{-3} = \dfrac{2\sin x}{(\cos x)^3}$.

Now, $\dfrac{2\sin x}{(\cos x)^3} = 2 \times \dfrac{\sin x}{\cos x} \times \dfrac{1}{(\cos x)^2} = 2\tan x \sec^2 x \therefore \dfrac{dy}{dx} = 2\tan x \sec^2 x$.

(c) $y = \dfrac{\ln(\mathrm{cosec}\, x)}{x} = \dfrac{\ln[(\sin x)^{-1}]}{x} = -\dfrac{\ln(\sin x)}{x} \therefore \dfrac{dy}{dx} = -\dfrac{\left(\frac{\cos x}{\sin x}\right) \times x - 1 \times \ln(\sin x)}{x^2}$

$$= -\frac{\frac{x\cos x - \sin x \ln(\sin x)}{\sin x}}{x^2}$$

$$= -\frac{x\cos x - \sin x \ln(\sin x)}{x^2 \sin x}$$

## An interesting result

A special case of the chain rule involves the case $y = x$. By viewing this as an application of the chain rule $\dfrac{dy}{dx} = \dfrac{dy}{du} \cdot \dfrac{du}{dx}$ we have (after setting $y = x$):

$$\frac{d(x)}{dx} = \frac{dx}{du} \cdot \frac{du}{dx} \Rightarrow 1 = \frac{dx}{du} \cdot \frac{du}{dx} \text{ i.e., } \frac{dx}{du} = 1 \Big/ \frac{du}{dx}$$

This **important** result is often written in the form

$$\frac{dy}{dx} = \frac{1}{\left(\frac{dx}{dy}\right)}$$

We find that this result is very useful with problems that deal with related rates.

# EXERCISES 19.3

**1.** Use the product rule to differentiate the following and then verify your answer by first expanding the brackets

(a) $(x^2+1)(2x-x^3+1)$ (b) $(x^3+x^2)(x^3+x^2-1)$

(c) $\left(\frac{1}{x^2}-1\right)\left(\frac{1}{x^2}+1\right)$ (d) $(x^3+x-1)(x^3+x+1)$

**2.** Use the quotient rule to differentiate the following

(a) $\frac{x+1}{x-1}$ (b) $\frac{x}{x+1}$ (c) $\frac{x+1}{x^2+1}$

(d) $\frac{x^2+1}{x^3-1}$ (e) $\frac{x^2}{2x+1}$ (f) $\frac{x}{1-2x}$

**3.** Differentiate the following

(a) $e^x\sin x$ (b) $x\log_e x$ (c) $e^x(2x^3+4x)$

(d) $x^4\cos x$ (e) $\sin x\cos x$ (f) $(1+x^2)\tan x$

(g) $\frac{4}{x^2}\times\sin x$ (h) $xe^x\sin x$ (i) $xe^x\log_e x$

**4.** Differentiate the following

(a) $\frac{x}{\sin x}$ (b) $\frac{\cos x}{x+1}$ (c) $\frac{e^x}{e^x+1}$

(d) $\frac{\sin x}{\sqrt{x}}$ (e) $\frac{x}{\log_e x}$ (f) $\frac{\log_e x}{x+1}$

(g) $\frac{e^x-1}{x+1}$ (h) $\frac{\sin x+\cos x}{\sin x-\cos x}$ (i) $\frac{x^2}{x+\log_e x}$

**5.** Differentiate the following

(a) $e^{-5x}+x$ (b) $\sin 4x-\frac{1}{2}\cos 6x$ (c) $e^{-\frac{1}{3}x}-\log_e(2x)+9x^2$

(d) $5\sin(5x)+3e^{2x}$ (e) $\tan(4x)+e^{2x}$ (f) $\cos(-4x)-e^{-3x}$

(g) $\log_e(4x+1)-x$ (h) $\log_e(e^{-x})+x$ (i) $\sin\left(\frac{x}{2}\right)+\cos(2x)$

(j) $\sin(7x-2)$ (k) $\sqrt{x}-\log_e(9x)$ (l) $\log_e(5x)-\cos(6x)$

**6.** Differentiate the following

(a) $\sin x^2+\sin^2 x$ (b) $\tan(2\theta)+\frac{1}{\sin\theta}$ (c) $\sin\sqrt{x}$

(d) $\cos\left(\frac{1}{x}\right)$ (e) $\cos^3\theta$ (f) $\sin(e^x)$

(g) $\tan(\log_e x)$ (h) $\sqrt{\cos(2x)}$ (i) $\cos(\sin\theta)$

(j) $4\sec\theta$ (k) $\operatorname{cosec}(5x)$ (l) $3\cot(2x)$

**7.** Differentiate the following

(a) $e^{2x+1}$ (b) $2e^{4-3x}$ (c) $2e^{4-3x^2}$ (d) $\sqrt{e^x}$

(e) $e^{\sqrt{x}}$ (f) $\frac{1}{2}e^{2x+4}$ (g) $\frac{1}{2}e^{2x^2+4}$ (h) $\frac{2}{e^{3x+1}}$

(i) $e^{3x^2-6x+1}$ (j) $e^{\sin(\theta)}$ (k) $e^{-\cos(2\theta)}$ (l) $e^{2\log_e(x)}$

(m) $\frac{2}{e^{-x}+1}$ (n) $(e^x - e^{-x})^3$ (o) $\sqrt{e^{2x+4}}$ (p) $e^{-x^2+9x-2}$

**8.** Differentiate the following

(a) $\log_e(x^2+1)$ (b) $\log_e(\sin\theta+\theta)$ (c) $\log_e(e^x - e^{-x})$

(d) $\log_e\left(\frac{1}{x+1}\right)$ (e) $(\log_e x)^3$ (f) $\sqrt{\log_e x}$

(g) $\log_e(\sqrt{x-1})$ (h) $\log_e(1-x^3)$ (i) $\log_e\left(\frac{1}{\sqrt{x+2}}\right)$

(j) $\log_e(\cos^2 x+1)$ (k) $\log_e(x\sin x)$ (l) $\log_e\left(\frac{x}{\cos x}\right)$

**9.** Differentiate the following

(a) $x\log_e(x^3+2)$ (b) $\sqrt{x}\sin^2 x$ (c) $\cos^2\sqrt{\theta}$

(d) $x^3e^{-2x^2+3}$ (e) $\cos(x\log_e x)$ (f) $\log_e(\log_e x)$

(g) $\frac{x^2-4x}{\sin(x^2)}$ (h) $\frac{10x+1}{\log_e(10x+1)}$ (i) $\frac{\cos(2x)}{e^{1-x}}$

(j) $x^2\log_e(\sin 4x)$ (k) $e^{-\sqrt{x}}\sin\sqrt{x}$ (l) $\cos(2x\sin x)$

(m) $\frac{e^{5x+2}}{1-4x}$ (n) $\frac{\log_e(\sin\theta)}{\cos\theta}$ (o) $\frac{x}{\sqrt{x+1}}$

(p) $x\sqrt{x^2+2}$ (q) $(x^3+x)\sqrt[3]{x+1}$ (r) $(x^3-1)\sqrt{x^3+1}$

(s) $\frac{1}{x}\log_e(x^2+1)$ (t) $\log_e\left(\frac{x^2}{x^2+2x}\right)$ (u) $\frac{\sqrt{x-1}}{x}$

(v) $e^{-x}\sqrt{x^2+9}$ (w) $(8-x^3)\sqrt{2-x}$ (x) $x^n\ln(x^n-1)$

**10.** Find the value of $x$ where the function $x \mapsto xe^{-x}$ has a horizontal tangent.

**11.** Find the gradient of the function $x \mapsto \sin\left(\frac{1}{x}\right)$, where $x = \frac{2}{\pi}$.

**12.** Find the gradient of the function $x \mapsto \log_e(x^2 + 4)$ at the point where the function crosses the $y$–axis.

**13.** For what value(s) of $x$ will the function $x \mapsto \ln(x^2 + 1)$ have a gradient of 1.

**14.** Find the rate of change of the function $x \mapsto e^{-x^2 + 2}$ at the point $(1, e)$.

**15.** Find (a) $\frac{d}{dx}(\sin x \cos x)$ (b) $\frac{d}{dx}(\sin x°)$ (c) $\frac{d}{dx}(\cos x°)$

**16.** (a) If $y$ is the product of three functions, i.e., $y = f(x)g(x)h(x)$, show that $\frac{dy}{dx} = f'(x)g(x)h(x) + f(x)g'(x)h(x) + f(x)g(x)h'(x)$.

(b) Hence, differentiate the following
i. $x^2 \sin x \cos x$
ii. $e^{-x^3} \sin(2x) \log_e(\cos x)$

**17.** (a) Given that $f(x) = 1 - x^3$ and $g(x) = \log_e x$, find
i. $(f \circ g)'(x)$
ii. $(g \circ f)'(x)$

(b) Given that $f(x) = \sin(x^2)$ and $g(x) = e^{-x}$, find
i. $(f \circ g)'(x)$
ii. $(g \circ f)'(x)$

**18.** Given that $T(\theta) = \frac{\cos k\theta}{2 + 3\sin k\theta}$, $k \neq 0$, determine $T'\left(\frac{\pi}{2k}\right)$.

**19.** If $f(x) = (x - a)^m (x - b)^n$, find $x$ such that $f'(x) = 0$.

**20.** If $f(\theta) = \sin\theta^m \cos\theta^n$, find $\theta$ such that $f'(\theta) = 0$.

**21.** Differentiate the following

(a) $f(x) = \cot 4x$ (b) $g(x) = \sec 2x$ (c) $f(x) = \operatorname{cosec} 3x$

(d) $y = \sin\left(3x + \frac{\pi}{2}\right)$ (e) $y = \cot\left(\frac{\pi}{4} - x\right)$ (f) $y = \sec(2x - \pi)$

**22.** Differentiate the following

(a) $\sec x^2$ (b) $\sin x \sec x$ (c) $\ln(\sec x)$

(d) $\cot^3 x$ (e) $\frac{x}{\operatorname{cosec} x}$ (f) $\frac{\operatorname{cosec} x}{\sin x}$

(g) $x^4 \operatorname{cosec}(4x)$ (h) $\tan 2x \cot x$ (i) $\sqrt{\sec x + \cos x}$

**23.** Differentiate the following

(a) $e^{\sec x}$ (b) $\sec(e^x)$ (c) $e^x \sec x$

(d) $\cot(\ln x)$ (e) $\ln(\cot 5x)$ (f) $\cot x \ln x$

(g) $\operatorname{cosec}(\sin x)$ (h) $\sin(\operatorname{cosec} x)$ (i) $\sin x \operatorname{cosec} x$

**4.** (a) $\frac{\sin x - x\cos x}{\sin^2 x}$ (b) $\frac{-[\sin x(x+1)+\cos x]}{(x+1)^2}$ (c) $\frac{e^x}{(e^x+1)^2}$ (d) $\frac{2x\cos x - \sin x}{2x\sqrt{x}}$ (e) $\frac{\ln x - 1}{(\ln x)^2}$ (f) $\frac{(x+1) - x\ln x}{x(x+1)^2}$ (g) $\frac{xe^x+1}{(x+1)^2}$ (h) $\frac{-2}{(\sin x - \cos x)^2}$ (i) $\frac{x^2 - x + 2x\ln x}{(x+\ln x)^2}$ **5.** (a) $-5e^{-5x}+1$

(b) $4\cos 4x + 3\sin 6x$ (c) $-\frac{1}{3}e^{-\frac{1}{3}x} - \frac{1}{x} + 18x$ (d) $25\cos 5x + 6e^{2x}$ (e) $4\sec^2 4x + 2e^{2x}$

(f) $-4\sin(4x) + 3e^{-3x}$ (g) $\frac{4}{4x+1} - 1$ (h) 0 (i) $\frac{1}{2}\cos\left(\frac{x}{2}\right) - 2\sin 2x$ (j) $7\cos(7x-2)$ (k) $\frac{1}{2\sqrt{x}} - \frac{1}{x}$

(l) $\frac{1}{x} + 6\sin 6x$ **6.** (a) $2x\cos x^2 + 2\sin x\cos x$ (b) $2\sec^2 2\theta - \frac{\cos\theta}{\sin^2\theta}$ (c) $\frac{1}{2\sqrt{x}}\cos\sqrt{x}$ (d) $\frac{1}{x^2}\sin\left(\frac{1}{x}\right)$

(e) $-3\sin\theta\cdot\cos^2\theta$ (f) $e^x\cos(e^x)$ (g) $\frac{1}{x}\sec^2(\log_e x)$ (h) $\frac{-\sin 2x}{\sqrt{\cos 2x}}$ (i) $-\cos\theta\cdot\sin(\sin\theta)$

(j) $4\sin\theta\cdot\sec^2\theta$ (k) $-5\cos 5x\cdot\csc^2(5x)$ (l) $-6\csc^2(2x)$

**7.** (a) $2e^{2x+1}$ (b) $-6e^{4-3x}$ (c) $-12xe^{4-3x^2}$ (d) $\frac{1}{2}\sqrt{e^x}$ (e) $\frac{1}{2\sqrt{x}}e^{\sqrt{x}}$ (f) $e^{2x+4}$ (g) $2xe^{2x^2+4}$

(h) $-\frac{6}{e^{3x+1}}$ (i) $(6x-6)e^{3x^2-6x+1}$ (j) $\cos(\theta)e^{\sin\theta}$ (k) $2\sin(2\theta)e^{-\cos 2\theta}$ (l) $2x$ (m) $\frac{2e^{-x}}{(e^{-x}+1)^2}$

(n) $3(e^x+e^{-x})(e^x-e^{-x})^2$ (o) $e^{x+2}$ (p) $(-2x+9)e^{-x^2+9x-2}$ **8.** (a) $\frac{2x}{x^2+1}$ (b) $\frac{\cos\theta+1}{\sin\theta+\theta}$

(c) $\frac{e^x+e^{-x}}{e^x-e^{-x}}$ (d) $-\frac{1}{x+1}$ (e) $\frac{3}{x}(\ln x)^2$ (f) $\frac{1}{2x\sqrt{\ln x}}$ (g) $\frac{1}{2(x-1)}$ (h) $\frac{-3x^2}{1-x^3}$ (i) $-\frac{1}{2(x+2)}$

(j) $\frac{-2\sin x\cos x}{\cos^2 x+1}$ (k) $\frac{1}{x} + \cot x$ (l) $\frac{1}{x} + \tan x$

**9.** (a) $\ln(x^3+2) + \frac{3x^3}{x^3+2}$ (b) $\frac{\sin^2 x}{2\sqrt{x}} + 2\sqrt{x}\sin x\cos x$ (c) $-\frac{1}{\sqrt{\theta}}\sin\sqrt{\theta}\cdot\cos\sqrt{\theta}$

(d) $(3x^2-4x^4)e^{-2x^2+3}$ (e) $-(\ln x+1)\sin(x\ln x)$ (f) $\frac{1}{x\ln x}$

(g) $\frac{(2x-4)\cdot\sin(x^2) - 2x\cdot\cos(x^2)(x^2-4x)}{(\sin x^2)^2}$ (h) $\frac{10(\ln(10x+1)-1)}{[\ln(10x+1)]^2}$

(i) $(\cos 2x - 2\sin 2x)e^{x-1}$ (j) $2x\ln(\sin 4x) + 4x^2\cot 4x$ (k) $(\cos\sqrt{x} - \sin\sqrt{x})\frac{1}{2\sqrt{x}}e^{-\sqrt{x}}$

(l) $-(2\sin x + 2x\cos x)\cdot\sin(2x\sin x)$ (m) $\frac{e^{5x+2}(9-20x)}{(1-4x)^2}$ (n) $\frac{\cos^2\theta + \sin^2\theta\ln(\sin\theta)}{\sin\theta\cos^2\theta}$

(o) $\frac{x+2}{2(x+1)\sqrt{x+1}}$ (p) $\frac{2x^2+2}{\sqrt{x^2+2}}$ (q) $\frac{10x^3+9x^2+4x+3}{3(x+1)^{2/3}}$ (r) $\frac{3x^2(3x^3+1)}{2\sqrt{x^3+1}}$

(s) $\frac{2}{x^2+1} - \frac{1}{x^2}\ln(x^2+1)$ (t) $\frac{2}{x(x+2)}$ (u) $\frac{2-x}{2x^2\sqrt{x-1}}$ (v) $\frac{-x^2+x-9}{\sqrt{x^2+9}}\cdot e^{-x}$ (w) $\frac{7x^3-12x^2-8}{2\sqrt{2-x}}$

(x) $nx^{n-1}\ln(x^n-1) + \frac{nx^{2n-1}}{x^n-1}$ **10.** $x = 1$ **11.** 0 **12.** 0 **13.** 1 **14.** $-2e$ **15.** (a) $\cos^2 x - \sin^2 x$

(b) $\frac{\pi}{180}\cos x°$ (c) $-\frac{\pi}{180}\sin x°$ **16.** (b) i. $2x\sin x\cos x + x^2\cos^2 x - x^2\sin^2 x$



ii. $e^{-x^3}(2\cos 2x\ln\cos x - 3x^2\sin 2x\ln\cos x - \sin 2x\tan x)$ **17.** (a) i. $-\frac{3}{x}(\ln x)^2$ ii. $-\frac{3x^2}{1-x^3}$

(b) i. $-2e^{-2x}\cdot\cos(e^{-2x})$ ii. $-2x\cos x^2\cdot e^{-\sin x^2}$ **18.** $-\frac{1}{5}k$ **19.** $x = a, b, \frac{mb+na}{m+n}$

**20.** $\{\theta : n\tan\theta^m\cdot\tan\theta^n = m\theta^{m-n}\}$ **21.** (a) $-4\csc(4x)$ (b) $2\sec(2x)\tan(2x)$

(c) $3\cot(3x)\csc(3x)$ (d) $-3\sin(3x)$ (e) $\csc^2\left(\frac{\pi}{4}-x\right)$ (f) $-2\sec(2x)\tan(2x)$

**22.** (a) $2x\sec(x^2)\tan(x^2)$ (b) $\sec^2 x$ (c) $\tan x$ (d) $-3\cot^2 x\csc^2 x$ (e) $x\cos x + \sin x$

(f) $-2\cot x\csc^2 x$ (g) $4x^3\csc(4x) - 4x^4\cot(4x)\csc(4x)$ (h) $2\cot x\sec^2(2x) - \csc^2 x\tan(2x)$

(i) $\frac{\sec x\tan x - \sin x}{2\sqrt{\cos x + \sec x}}$ **23.** (a) $e^{\sec x}\sec x\tan x$ (b) $e^x\sec(e^x)\tan(e^x)$

(c) $e^x\sec(x) + e^x\sec(x)\tan(x)$ (d) $\frac{-\csc^2(\log x)}{x}$ (e) $-5\csc(5x)\sec(5x)$

(f) $\frac{\cot(x)}{x} - \csc^2(x)\log x$ (g) $-\cos x\cot(\sin x)\csc(\sin x)$ (h) $-\cos(\csc x)\cot x\csc x$ (i) 0

## EXERCISE 19.4

**1.** (a) $\frac{2}{4x^2+1}$ (b) $\frac{1}{\sqrt{9-x^2}}$ (c) $\frac{-2}{\sqrt{1-4x^2}}$ (d) $\frac{4}{\sqrt{1-16x^2}}$ (e) $\frac{2}{x^2+4}$ (f) $\frac{1}{\sqrt{2x-x^2}}$ (g) $\frac{-1}{\sqrt{16-x^2}}$

(h) $\frac{1}{\sqrt{4-(x+1)^2}}$ (i) $\frac{1}{(4-x)^2+1}$ (j) $\frac{-1}{\sqrt{4x-x^2}}$ (k) $\frac{6}{4x^2+9}$ (l) $\frac{-1}{\sqrt{-x^2+x+2}}$

**2.** (a) $\frac{2x}{x^4+1}$ (b) $\frac{1}{2\sqrt{x-x^2}}$ (c) $\frac{1}{2\sqrt{x^3-x^2}}$ (d) $\frac{-\sin x}{\sqrt{1-\cos^2 x}} = \begin{cases} -1 & \text{if } \sin x > 0 \\ 1 & \text{if } \sin x < 0 \end{cases}$ (e) $\frac{1}{2x\sqrt{x-1}}$

(f) $\frac{1}{\sqrt{1-x^2}\,\mathrm{Sin}^{-1}x}$ (g) $\frac{e^x}{1+e^{2x}}$ (h) $\frac{1}{\sqrt{e^{2x}-1}}$ (i) $\frac{e^{\arcsin x}}{\sqrt{1-x^2}}$ (j) $\frac{-4}{(4x^2+1)[\tan^{-1}(2x)]^2}$

(k) $\frac{-1}{\sqrt{1-x^2}(\sin^{-1}(x))^{3/2}}$ (l) $\frac{2}{\sqrt{1-x^2}(\cos^{-1}(x))^3}$ (m) $\frac{-4x}{\sqrt{1-4x^2}}$ (n) $\frac{-4x}{\sqrt{1-4x^2}}$ (o) $\frac{-1}{x^2\sqrt{1-x^2}}$

**3.** (a) $\mathrm{Tan}^{-1}x + \frac{x}{1+x^2}$ (b) $\frac{x - \sqrt{1-x^2}\sin^{-1}x}{x^2\sqrt{1-x^2}}$ (c) $\frac{x + \sqrt{1-x^2}\cos^{-1}x}{(\cos^{-1}x)^2\sqrt{1-x^2}}$

(d) $\frac{-2x^2\tan^{-1}x + x - 2\tan^{-1}x}{x^3(x^2+1)}$ (e) $\frac{2x^2\log x + \sqrt{1-x^4}\sin^{-1}(x^2)}{x\sqrt{1-x^4}}$ (f) $\frac{-\sqrt{1-x}\cos^{-1}\sqrt{x} - \sqrt{x}}{2x^{3/2}\sqrt{1-x}}$

(g) $e^x\tan^{-1}(e^x) + \frac{e^{2x}}{1+e^{2x}}$ (h) $2x\tan^{-1}\left(\frac{x}{2}\right) + 2$ (i) $1 - \frac{x}{\sqrt{4-x^2}}\mathrm{Sin}^{-1}\left(\frac{x}{2}\right)$ **4.** $0, k = \frac{\pi}{2}$

**6.** (b) $k = \frac{\pi}{2}$

**7.** (a) $f'(x) = \frac{-\pi}{x\sqrt{x^2-\pi^2}}, x > \pi$ and $\frac{\pi}{x\sqrt{x^2-\pi^2}}, x < -\pi$ ; $\mathrm{dom}(f) = ]-\infty, -\pi[ \cup ]\pi, \infty[$

(b) $f'(x) = \frac{1}{x\sqrt{2x-1}}, x > \frac{1}{2}$ ; $\mathrm{dom}(f') = \left]\frac{1}{2}, \infty\right[$ , $\mathrm{dom}(f) = \left[\frac{1}{2}, \infty\right[$

(c) $f'(x) = \frac{1}{\sqrt{1-x^2}}\mathrm{Cos}^{-1}\left(\frac{x}{2}\right) - \frac{1}{\sqrt{4-x^2}}\mathrm{Sin}^{-1}(x), -1 < x < 1$ ; $\mathrm{dom}(f) = [-1, 1]$