Integration - practice questions

The presentation is structured as follows. You're given an integral. You should try and solve it.

The presentation is structured as follows. You're given an integral. You should try and solve it. If you struggle, then there'll be a hint - usually an indication of the method you should use.

The presentation is structured as follows. You're given an integral. You should try and solve it. If you struggle, then there'll be a hint - usually an indication of the method you should use. Finally a full solution will be given.

Example 1 - easy

Find $\int x\sqrt{x^2+1}\,dx$

Example 1 - easy

Find $\int x\sqrt{x^2+1}\, dx$

Hint:

Example 1 - easy

Find $\int x\sqrt{x^2+1}\,dx$

Hint: use substitution $u = x^2 + 1$.

Example 1 - easy

Find $\int x\sqrt{x^2+1}\,dx$

Hint: use substitution $u = x^2 + 1$.

Solution:

Example 1 - easy

Find $\int x\sqrt{x^2+1}\,dx$

Hint: use substitution $u = x^2 + 1$.

Solution: If $u = x^2 + 1$, then $\frac{du}{dx} = 2x$, which gives $dx = \frac{1}{2x}\,du$.
The integral becomes:

$$\int x\sqrt{x^2+1}\,dx = \int x\sqrt{u}\frac{1}{2x}\,du = \frac{1}{2}\int \sqrt{u}\,du =$$
$$= \frac{1}{3}u^{\frac{3}{2}} + c = \frac{1}{3}u\sqrt{u} + c = \frac{1}{3}(x^2+1)\sqrt{x^2+1} + c$$

Example 2 - easy

Find $\int x\sqrt{3x-1}\,dx$

Example 2 - easy

Find $\int x\sqrt{3x-1}\,dx$

Hint:

Example 2 - easy

Find $\int x\sqrt{3x-1}\,dx$

Hint: use substitution $u = 3x - 1$.

Example 2 - easy

Find $\int x\sqrt{3x-1}\,dx$

Hint: use substitution $u = 3x - 1$.

Solution:

Example 2 - easy

Find $\int x\sqrt{3x-1}\,dx$

Hint: use substitution $u = 3x - 1$.

Solution: If $u = 3x - 1$, then $\frac{du}{dx} = 3$, which gives $dx = \frac{1}{3}\,du$.
We also have $x = \frac{u+1}{3}$.
The integral becomes:

$$\int x\sqrt{3x-1}\,dx = \int \frac{u+1}{3}\sqrt{u}\frac{1}{3}\,du = \frac{1}{9}\int u\sqrt{u} + \sqrt{u}\,du =$$
$$= \frac{2}{45}u^{\frac{5}{2}} + \frac{2}{27}u^{\frac{3}{2}} + c = \frac{2}{45}(3x-1)^{\frac{5}{2}} + \frac{2}{27}(3x-1)^{\frac{3}{2}} + c$$

Example 3 - easy

Find $\int x^4 \ln x \; dx$

Example 3 - easy

Find $\int x^4 \ln x \, dx$

Hint:

Example 3 - easy

Find $\int x^4 \ln x \, dx$

Hint: use integration by parts with $f = \ln x$ and $g' = x^4$.

Example 3 - easy

Find $\int x^4 \ln x \, dx$

Hint: use integration by parts with $f = \ln x$ and $g' = x^4$.

Solution:

Example 3 - easy

Find $\int x^4 \ln x \, dx$

Hint: use integration by parts with $f = \ln x$ and $g' = x^4$.

Solution: If $f = \ln x$, then $f' = \frac{1}{x}$. Also if $g' = x^4$, then $g = \frac{1}{5}x^5$.
The integral becomes:

$$\int x^4 \ln x \, dx = \frac{1}{5}x^5 \ln x - \int \frac{1}{x} \times \frac{1}{5}x^5 \, dx = \frac{1}{5}x^5 \ln x - \frac{1}{5}\int x^4 \, dx =$$
$$= \frac{1}{5}x^5 \ln x - \frac{1}{25}x^5 + c$$

Example 4 - easy

Find $\int \frac{x}{x^2 - 1}\, dx$

Example 4 - easy

Find $\int \frac{x}{x^2 - 1}\, dx$

Hint:

Example 4 - easy

Find $\int \frac{x}{x^2 - 1}\, dx$

Hint: the denominator can be factorized, so you can try partial fractions, but it's much better to look for the derivative of the denominator in the numerator.

Example 4 - easy

Find $\int \frac{x}{x^2 - 1}\, dx$

Hint: the denominator can be factorized, so you can try partial fractions, but it's much better to look for the derivative of the denominator in the numerator.

Solution:

Example 4 - easy

Find $\int \frac{x}{x^2 - 1}\, dx$

Hint: the denominator can be factorized, so you can try partial fractions, but it's much better to look for the derivative of the denominator in the numerator.

Solution: the derivative of the denominator is $2x$, so this is what we want in the numerator:

$$\int \frac{x}{x^2-1}\, dx = \frac{1}{2}\int \frac{2x}{x^2-1}\, dx = \frac{1}{2}\ln|x^2-1| + c$$

Example 5 - easy

Find $\int \frac{4x-5}{6x^2-x-2}\,dx$

Example 5 - easy

Find $\int \frac{4x-5}{6x^2-x-2}\,dx$

Hint:

Example 5 - easy

Find $\int \frac{4x-5}{6x^2-x-2}\, dx$

Hint: we can factorize the denominator, so partial fractions can be used.

Example 5 - easy

Find $\int \frac{4x-5}{6x^2-x-2}\,dx$

Hint: we can factorize the denominator, so partial fractions can be used.

Solution:

Example 5 - easy

Find $\int \frac{4x-5}{6x^2-x-2}\, dx$

Hint: we can factorize the denominator, so partial fractions can be used.

Solution: $6x^2 - x - 2 = (3x-2)(2x+1)$, so we write:

$$\frac{4x-5}{6x^2-x-2} = \frac{A}{3x-2} + \frac{B}{2x+1} = \frac{A(2x+1)+B(3x-2)}{(3x-2)(2x+1)}$$

Equating the coefficients of x in the numerator gives the following system of equations:

$$\begin{cases} 2A + 3B = 4 \\ A - 2B = -5 \end{cases}$$

Solving the above gives $A = -1$ and $B = 2$.

Example 5 continued

$$\int \frac{4x-5}{6x^2-x-2}\,dx = \int \frac{-1}{3x-2} + \frac{2}{2x+1}\,dx =$$

$$= -\int \frac{1}{3x-2}\,dx + 2\int \frac{1}{2x+1}\,dx = -\frac{1}{3}\ln|3x-2| + \ln|2x+1| + c$$

Example 6 - easy

Find $\displaystyle\int \frac{3}{x^2+2x+6}\,dx$

Example 6 - easy

Find $\int \frac{3}{x^2+2x+6}\,dx$

Hint:

Example 6 - easy

Find $\displaystyle\int \frac{3}{x^2+2x+6}\,dx$

Hint: the denominator cannot be factorized, so we complete the square and look for $\arctan x$ formula.

Example 6 - easy

Find $\int \frac{3}{x^2+2x+6}\,dx$

Hint: the denominator cannot be factorized, so we complete the square and look for $\arctan x$ formula.

Solution:

Example 6 - easy

Find $\int \frac{3}{x^2+2x+6}\,dx$

Hint: the denominator cannot be factorized, so we complete the square and look for $\arctan x$ formula.

Solution:

$$\int \frac{3}{x^2+2x+6}\,dx = \int \frac{3}{(x+1)^2+5}\,dx = \frac{3}{\sqrt{5}}\int \frac{\sqrt{5}}{(x+1)^2+5}\,dx =$$
$$= \frac{3}{\sqrt{5}}\arctan\left(\frac{x+1}{\sqrt{5}}\right)+c$$

Example 7 - easy

Find $\int x^2 e^{3x}\, dx$

Example 7 - easy

Find $\int x^2 e^{3x}\, dx$

Hint:

Example 7 - easy

Find $\int x^2 e^{3x}\, dx$

Hint: you need to integrate by parts twice.

Example 7 - easy

Find $\int x^2 e^{3x} \, dx$

Hint: you need to integrate by parts twice.

Solution:

Example 7 - easy

Find $\int x^2 e^{3x}\, dx$

Hint: you need to integrate by parts twice.

Solution: we let $f = x^2$ and $g' = e^{3x}$ first. Then we let $f_2 = x$ and $g_2' = e^{3x}$.

$$\int x^2 e^{3x}\, dx = \frac{1}{3}x^2 e^{3x} - \frac{2}{3}\int x e^{3x}\, dx = \frac{1}{3}x^2 e^{3x} - \frac{2}{9}x e^{3x} + \frac{2}{9}\int e^{3x}\, dx =$$
$$= \frac{1}{3}x^2 e^{3x} - \frac{2}{9}x e^{3x} + \frac{2}{27}e^{3x} + c$$

Example 8 - easy

Find $\int e^{2x} \sin(x)\ dx$

Example 8 - easy

Find $\displaystyle\int e^{2x} \sin(x)\, dx$

Hint:

Example 8 - easy

Find $\int e^{2x} \sin(x) \, dx$

Hint: you need to integrate by parts twice and use the resulting equation to find the integral.

Example 8 - easy

Find $\int e^{2x} \sin(x)\, dx$

Hint: you need to integrate by parts twice and use the resulting equation to find the integral.

Solution:

Example 8 - easy

Find $\int e^{2x} \sin(x)\, dx$

Hint: you need to integrate by parts twice and use the resulting equation to find the integral.

Solution: we let $f = e^{2x}$ and $g' = \sin x$ first. Then we let $f_2 = e^{2x}$ and $g_2' = \cos x$.

$$\int e^{2x} \sin(x)\, dx = -e^{2x} \cos x + 2 \int e^{2x} \cos(x)\, dx =$$

$$= -e^{2x} \cos x + 2e^{2x} \sin x - 4 \int e^{2x} \sin(x)\, dx$$

Example 8 continued

So we get that:

$$\int e^{2x}\sin(x)\,dx = -e^{2x}\cos x + 2e^{2x}\sin x - 4\int e^{2x}\sin(x)\,dx$$

Rearranging the equation we got and dividing by 5 gives:

$$\int e^{2x}\sin(x)\,dx = \frac{-e^{2x}\cos x + 2e^{2x}\sin x}{5} + c$$

Example 9 - moderate

Find $\displaystyle\int \frac{1}{e^x + 9e^{-x}}\, dx$

Example 9 - moderate

Find $\int \frac{1}{e^x + 9e^{-x}} \, dx$

Hint:

Example 9 - moderate

Find $\int \frac{1}{e^x + 9e^{-x}}\, dx$

Hint: try substitution $u = e^x$ and look for $\arctan x$.

Example 9 - moderate

Find $\int \frac{1}{e^x + 9e^{-x}}\ dx$

Hint: try substitution $u = e^x$ and look for $\arctan x$.

Solution:

Example 9 - moderate

Find $\int \frac{1}{e^x + 9e^{-x}}\, dx$

Hint: try substitution $u = e^x$ and look for $\arctan x$.

Solution: if $u = e^x$, then $\frac{du}{dx} = e^x$, so $dx = \frac{1}{e^x} du = \frac{1}{u} du$.

$$\int \frac{1}{e^x + 9e^{-x}}\, dx = \int \frac{1}{u + 9u^{-1}} \times \frac{1}{u}\, du = \int \frac{1}{u^2 + 9}\, du =$$
$$= \frac{1}{3} \int \frac{3}{u^2 + 9}\, du = \frac{1}{3} \arctan\left(\frac{u}{3}\right) + c =$$
$$= \frac{1}{3} \arctan\left(\frac{e^x}{3}\right) + c$$

Example 10 - moderate

Find $\int \frac{x^2}{1-x^2}\, dx$

Example 10 - moderate

Find $\int \frac{x^2}{1-x^2}\,dx$

Hint:

Example 10 - moderate

Find $\int \frac{x^2}{1-x^2}\, dx$

Hint: divide and then use partial fractions.

Example 10 - moderate

Find $\int \frac{x^2}{1-x^2}\, dx$

Hint: divide and then use partial fractions.

Solution:

Example 10 - moderate

Find $\int \frac{x^2}{1-x^2}\, dx$

Hint: divide and then use partial fractions.

Solution:

$$\int \frac{x^2}{1-x^2}\, dx = \int \frac{x^2-1+1}{1-x^2}\, dx = \int -1 + \frac{1}{1-x^2}\, dx =$$
$$= \int -1 + \frac{\frac{1}{2}}{1-x} + \frac{\frac{1}{2}}{1+x}\, dx =$$
$$= -x - \frac{1}{2}\ln|1-x| + \frac{1}{2}\ln|1+x| + c$$

Example 11 - moderate

Find $\int \sqrt{25 - x^2}\, dx$

Example 11 - moderate

Find $\int \sqrt{25 - x^2}\, dx$

Hint:

Example 11 - moderate

Find $\int \sqrt{25 - x^2}\, dx$

Hint: we have the form $\sqrt{a^2 - x^2}$, so we're hoping for $\arcsin x$. Use the substitution $x = 5\sin u$. Then remember to use double angle identity.

Example 11 - moderate

Find $\int \sqrt{25 - x^2}\, dx$

Hint: we have the form $\sqrt{a^2 - x^2}$, so we're hoping for $\arcsin x$. Use the substitution $x = 5 \sin u$. Then remember to use double angle identity.

Solution:

Example 11 - moderate

Find $\int \sqrt{25 - x^2}\, dx$

Hint: we have the form $\sqrt{a^2 - x^2}$, so we're hoping for $\arcsin x$. Use the substitution $x = 5\sin u$. Then remember to use double angle identity.

Solution:It $x = 5\sin u$, then $\frac{dx}{du} = 5\cos u$, so $dx = 5\cos u\ du$.

$$\int \sqrt{25 - x^2}\, dx = \int \sqrt{25 - 25\sin^2 u} \times 5\cos u\ du = 25 \int \cos^2 u\ du =$$

$$= 25 \int \frac{1 + \cos 2u}{2}\ du = \frac{25}{2}u + \frac{25}{4}\sin 2u + c$$

Example 11 - moderate

So we got:

$$\int \sqrt{25 - x^2}\, dx = \frac{25}{2}u + \frac{25}{4}\sin 2u + c$$

where $x = 5\sin u$ or $\sin u = \dfrac{x}{5}$.

Example 11 - moderate

So we got:

$$\int \sqrt{25 - x^2}\, dx = \frac{25}{2}u + \frac{25}{4}\sin 2u + c$$

where $x = 5\sin u$ or $\sin u = \dfrac{x}{5}$. We of course have $u = \arcsin \dfrac{x}{5}$.

Example 11 - moderate

So we got:

$$\int \sqrt{25 - x^2}\, dx = \frac{25}{2}u + \frac{25}{4}\sin 2u + c$$

where $x = 5\sin u$ or $\sin u = \frac{x}{5}$. We of course have $u = \arcsin \frac{x}{5}$. What about $\sin 2u$?

Example 11 - moderate

So we got:

$$\int \sqrt{25-x^2}\,dx = \frac{25}{2}u + \frac{25}{4}\sin 2u + c$$

where $x = 5\sin u$ or $\sin u = \frac{x}{5}$. We of course have $u = \arcsin\frac{x}{5}$. What about $\sin 2u$? Here it is useful to draw an appropriate triangle with an angle u and sides x and 5 - the remaining side is then $\sqrt{25-x^2}$.

Example 11 - moderate

So we got:

$$\int \sqrt{25 - x^2}\, dx = \frac{25}{2}u + \frac{25}{4}\sin 2u + c$$

where $x = 5\sin u$ or $\sin u = \frac{x}{5}$. We of course have $u = \arcsin\frac{x}{5}$. What about $\sin 2u$? Here it is useful to draw an appropriate triangle with an angle u and sides x and 5 - the remaining side is then $\sqrt{25 - x^2}$. We then have $\sin 2u = 2\sin u \cos u = 2 \times \frac{x}{5} \times \frac{\sqrt{25-x^2}}{5}$.

Example 11 - moderate

So we got:

$$\int \sqrt{25 - x^2}\, dx = \frac{25}{2}u + \frac{25}{4}\sin 2u + c$$

where $x = 5\sin u$ or $\sin u = \dfrac{x}{5}$. We of course have $u = \arcsin \dfrac{x}{5}$. What about $\sin 2u$? Here it is useful to draw an appropriate triangle with an angle u and sides x and 5 - the remaining side is then $\sqrt{25 - x^2}$. We then have $\sin 2u = 2\sin u \cos u = 2 \times \frac{x}{5} \times \frac{\sqrt{25-x^2}}{5}$.
Finally the integral becomes:

$$\int \sqrt{25 - x^2}\, dx = \frac{25}{2}\arcsin \frac{x}{5} + \frac{x}{2}\sqrt{25 - x^2} + c$$

Example 12 - easy/moderate

Find $\int x\sqrt{25-x^2}\,dx$

Example 12 - easy/moderate

Find $\int x\sqrt{25 - x^2}\, dx$

Hint:

Example 12 - easy/moderate

Find $\int x\sqrt{25-x^2}\,dx$

Hint: we may be tempted to set $x = 5\sin u$ again, but there is of course a much better way. We will do it the hard way anyway to see that it also gives the correct result.

Example 12 - easy/moderate

Find $\int x\sqrt{25-x^2}\,dx$

Hint: we may be tempted to set $x = 5\sin u$ again, but there is of course a much better way. We will do it the hard way anyway to see that it also gives the correct result.

Solution:

Example 12 - easy/moderate

Find $\int x\sqrt{25-x^2}\,dx$

Hint: we may be tempted to set $x = 5\sin u$ again, but there is of course a much better way. We will do it the hard way anyway to see that it also gives the correct result.

Solution: It $x = 5\sin u$, then $\frac{dx}{du} = 5\cos u$, so $dx = 5\cos u\ du$.

$$\int x\sqrt{25-x^2}\,dx = \int 5\sin u\sqrt{25-25\sin^2 u} \times 5\cos u\ du =$$
$$= 125\int \sin u\cos^2 u\ du$$

Now let $t = \cos u$, so $\frac{dt}{du} = -\sin u$, so $du = -\frac{1}{\sin u}\ dt$.

Example 12 - moderate

$$\int x\sqrt{25-x^2}\,dx = \int 5\sin u\sqrt{25-25\sin^2 u}\times 5\cos u\,du =$$

$$= 125\int \sin u\cos^2 u\,du =$$

$$= -125\int t^2\,dt = -\frac{125}{3}t^3 + c = -\frac{125}{3}\cos^3 u + c$$

We need to get back to x, we can do that again by drawing an appropriate triangle. It gives $\cos u = \frac{\sqrt{25-x^2}}{5}$. So:

$$\int x\sqrt{25-x^2}\,dx = -\frac{1}{3}(25-x^2)\sqrt{25-x^2} + c$$

Example 12 - again

Find $\int x\sqrt{25 - x^2}\, dx$

Example 12 - again

Find $\int x\sqrt{25-x^2}\,dx$

Hint:

Example 12 - again

Find $\int x\sqrt{25-x^2}\,dx$

Hint: it's of course much better to just let $u = 25 - x^2$, because then x will cancel.

Example 12 - again

Find $\int x\sqrt{25-x^2}\,dx$

Hint: it's of course much better to just let $u = 25 - x^2$, because then x will cancel.

Solution:

Example 12 - again

Find $\int x\sqrt{25-x^2}\,dx$

Hint: it's of course much better to just let $u = 25 - x^2$, because then x will cancel.

Solution: It $u = 25 - x^2$, then $\frac{du}{dx} = -2x$, so $dx = -\frac{1}{2x}\,du$.

$$\int x\sqrt{25-x^2}\,dx = \int x\sqrt{u} \times -\frac{1}{2x}\,du = -\frac{1}{2}\int \sqrt{u}\,du =$$
$$= -\frac{1}{3}u^{\frac{3}{2}} + c = -\frac{1}{3}(25-x^2)\sqrt{25-x^2} + c$$

Example 13 - hard

(a) Find the derivative of $\sec x + \tan x$.

Example 13 - hard

(a) Find the derivative of $\sec x + \tan x$.

(b) Find $\displaystyle\int \sec x \, dx$ by writing $\sec x$ as $\sec x \times \dfrac{\tan x + \sec x}{\sec x + \tan x}$.

Example 13 - hard

(a) Find the derivative of $\sec x + \tan x$.

(b) Find $\int \sec x \, dx$ by writing $\sec x$ as $\sec x \times \dfrac{\tan x + \sec x}{\sec x + \tan x}$.

(c) Find $\int \dfrac{1}{\sqrt{1+x^2}} \, dx$ by letting $x = \tan\theta$.

Example 13 - hard

(a) $(\sec x + \tan x)' = \sec x \tan x + \sec^2 x$.

Example 13 - hard

(a) $(\sec x + \tan x)' = \sec x \tan x + \sec^2 x$.

(b)

$$\int \sec x \; dx = \int \sec x \times \frac{\tan x + \sec x}{\sec x + \tan x} \; dx =$$

$$= \int \frac{\sec x \tan x + \sec^2 x}{\sec x + \tan x} \; dx = \ln|\sec x + \tan x| + c$$

Example 13 - hard

(c) If $x = \tan\theta$, then $\dfrac{dx}{d\theta} = \sec^2\theta$, so $dx = \sec^2\theta d\theta$.

$$\int \frac{1}{\sqrt{1+x^2}}\,dx = \int \frac{1}{\sqrt{1+\tan^2\theta}} \times \sec^2\theta\ d\theta =$$
$$= \int \sec\theta\ d\theta = \ln|\sec\theta + \tan\theta| + c$$

Now we need to change back to x. Since $x = \tan\theta$, then we draw a triangle with sides x and 1, the remaining side is $\sqrt{x^2+1}$. This gives $\sec\theta = \sqrt{x^2+1}$. So finally:

$$\int \frac{1}{\sqrt{1+x^2}}\,dx = \ln|\sqrt{x^2+1} + x| + c$$

The plan has changed, so the test will be on Monday. If there are any questions you can email me at T.J.Lechowski@gmail.com.