Complex numbers - revision [104 marks]

1. [Maximum mark: 5] EXN.1.AHL.TZ0.14

   (a)   Write down $2 + 5\mathrm{i}$ in exponential form. [2]

> **Markscheme**
>
> *This sample question was produced by experienced DP mathematics senior examiners to aid teachers in preparing for external assessment in the new MAA course. There may be minor differences in formatting compared to formal exam papers.
>
> $5.385\ldots\mathrm{e}^{1.1902\ldots\mathrm{i}} \approx 5.39\mathrm{e}^{1.19\mathrm{i}}$ **A1A1**
>
> **Note:** Accept equivalent answers: $5.39\mathrm{e}^{-5.09\mathrm{i}}$
>
> **[2 marks]**

    (b)
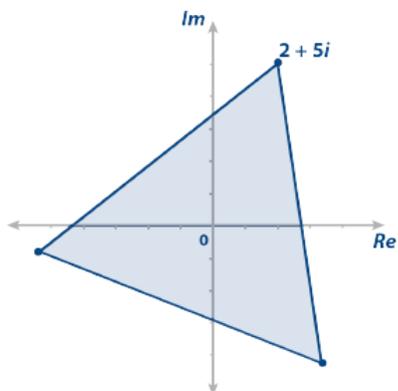
        An equilateral triangle is to be drawn on the Argand plane with one of the vertices at the point corresponding to $2 + 5\mathrm{i}$ and all the vertices equidistant from 0.

        Find the points that correspond to the other two vertices. Give your answers in Cartesian form. [3]

> **Markscheme**
>
> multiply by $\mathrm{e}^{\frac{2\pi}{3}\mathrm{i}}$ **(M1)**
>
> $-5.33 - 0.77\mathrm{i},\ 3.33 - 4.23\mathrm{i}$ **A1A1**
>
> **[3 marks]**

2. [Maximum mark: 6] 24M.1.AHL.TZ1.13

    Let $z_1 = 4 + 5\mathrm{i}$.

(a.i)   Find $|z_1|$.   [1]

$$\left(|z_1| = \sqrt{4^2 + 5^2} =\right) \ 6.40 \ \left(6.40312\ldots, \ \sqrt{41}\right) \quad \textbf{A1}$$

[1 mark]

(a.ii)   Find $\arg(z_1)$.   [1]

$$(\arg(z_1) =) \ 0.896 \ \left(0.896055\ldots, \ 51.3401\ldots^{\circ}, \ \arctan\left(\tfrac{5}{4}\right)\right) \quad \textbf{A1}$$

[1 mark]

Let $z_2 = 3e^{2i}$.

(b)   Find the area of the triangle on an Argand diagram with vertices $0, \ z_1$ and z_2.   [4]

angle in triangle is $2 - 0.896055\ldots$ **OR** $114.591^{\circ} - 51.3401\ldots^{\circ}$ *(A1)*

use of area of triangle formula   *(M1)*

$\tfrac{1}{2} \times 6.40312\ldots \times 3 \times \sin(2 - 0.896055\ldots)$ *(A1)*

$8.58 \ (8.57688\ldots)$ *A1*

Note: Accept methods that use Cartesian form or vector product.

[4 marks]

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3.   [Maximum mark: 6]   23N.1.AHL.TZ0.8

Given $z = \sqrt{3} - i$.

(a)   Write z in the form $z = re^{\theta i}$, where $r \in \mathbb{R}^{+}, -\pi < \theta \leq \pi$.   [2]

$$z = 2e^{-0.524i} \ \left(= 2e^{-\frac{\pi}{6}i}\right) \qquad \textbf{A1A1}$$

Note: Award *A1* for the correct modulus and *A1* for the correct argument.

[2 marks]

Let $z_1 = e^{2ti}$ and $z_2 = 2e^{\left(2t - \frac{\pi}{6}\right)i}$.

(b)   Find $\text{Im}\,(z_1 + z_2)$ in the form $p\,\sin\,(2t + q)$, where $p > 0$, $t \in \mathbb{R}$ and $-\pi \le q \le \pi$.   [4]

Markscheme

METHOD 1

$$z_1 + z_2 = e^{2ti}\left(1 + 2e^{-\frac{\pi}{6}i}\right) \qquad \text{(M1)}$$

$$= e^{2ti} \times 2.90931\ldots e^{-0.350879\ldots i} \qquad \text{(A1)}$$

$$\text{Im}\,(z_1 + z_2) = 2.91\,\sin\,(2t - 0.351) \qquad \text{A1A1}$$

METHOD 2

attempt to find the modulus of $\left(1 + 2e^{-\frac{\pi}{6}i}\right)$ (M1)

$$p = \left|1 + 2e^{-\frac{\pi}{6}i}\right|$$

attempt to find the argument of $\left(1 + 2e^{-\frac{\pi}{6}i}\right)$ (M1)

$$q = \arg\left(1 + 2e^{-\frac{\pi}{6}i}\right)$$

$$\text{Im}\,(z_1 + z_2) = 2.91\,\sin\,(2t - 0.351) \qquad \text{A1A1}$$

METHOD 3

sketching $\text{Im}\,(z_1 + z_2)$ (M1)

max is $(0.961,\ 2.91)$ (A1)

first root is 0.1754 (A1)

$$\text{Im}\,(z_1 + z_2) = 2.91\,\sin\,(2t - 0.351) \qquad \text{A1}$$

[4 marks]

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4.   [Maximum mark: 5]

Two AC (alternating current) electrical sources with the same frequencies are combined. The voltages from these sources can be expressed as $V_1 = 6\,\sin\,(at + 30°)$ and $V_2 = 6\,\sin\,(at + 90°)$.

The combined total voltage can be expressed in the form $V_1 + V_2 = V\,\sin\,(at + \theta°)$.

Determine the value of V and the value of θ.   [5]

METHOD 1 Analytical approach

attempt to express V_1 or V_2 in exponential form   *(M1)*

e.g. $V_1 = \mathrm{Im}\left(6\mathrm{e}^{\mathrm{i}\left(at+\frac{\pi}{6}\right)}\right)$, $V_2 = \mathrm{Im}\left(6\mathrm{e}^{\mathrm{i}\left(at+\frac{\pi}{2}\right)}\right)$

Note: Accept angles in radians or degrees.

$\left(V_1 + V_2 =\right)\ 6\mathrm{e}^{\mathrm{i}\times\frac{\pi}{6}} + 6\mathrm{e}^{\mathrm{i}\times\frac{\pi}{2}}$ *(A1)*

Note: This mark can be awarded even if seen as part of a correct larger expression.

$= 10.\,4\mathrm{e}^{1.05\mathrm{i}}\left(6\sqrt{3}\mathrm{e}^{\frac{\mathrm{i}\pi}{3}}\right)$ *(A1)*

so V is $10.\,4\left(10.\,3923\ldots,\ 6\sqrt{3}\right)$ and θ is 60 (degrees)   *A1A1*
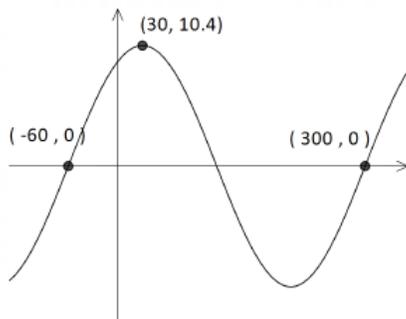
Note: Accept any value for θ that rounds to a 2sf answer of 60.

Do **not** accept a final answer for an angle in radians.

Do **not** award *A1* for answer of $60\,^{\circ}$ resulting from incorrect working.

METHOD 2 Graphical approach

let $at = x$ and plot $V_1 + V_2$ curves on GDC   *(M1)*
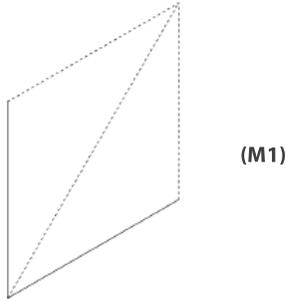

attempt to find maximum   *(M1)*

$V = 10.\,4$ *A1*

attempt to find any x-axis intercept (either -60 or 300)   *(M1)*

$\theta = 60$ (degrees) $\theta = -300$ (degrees)   **A1**

METHOD 3 Geometric approach

considering the rhombus

     (M1)

$V = \sqrt{6^2 + 6^2 - 2 \times 6 \times 6 \cos 120^\circ}$ **(M1)**

$\left(= \sqrt{108} = 6\sqrt{3}\right) = 10.4 \left(10.3923\ldots\right)$ **A1**

$\theta = 60$ (degrees)   **A2**

Note: An answer of $\theta = -300.$ is most likely to be seen in METHOD 2, but should be condoned in METHODS 1 and 3 if seen there.

[5 marks]

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5. [Maximum mark: 6]   23M.1.AHL.TZ2.11
Two AC (alternating current) electrical sources of equal frequencies are combined.

The voltage of the first source is modelled by the equation $V = 30 \sin (t + 60^\circ)$.

The voltage of the second source is modelled by the equation $V = 60 \sin (t + 10^\circ)$.

(a)   Determine the maximum voltage of the combined sources.   [2]

Markscheme

$30 \sin (t + 60^\circ) + 60 \sin (t + 10^\circ)$ **(M1)**

finding maximum graphically

$82.5 \text{ (V)} \ (82.5471\ldots)$ **A1**

Note: Award *M1A0* for 83.

[2 marks]

(b) Using your graphic display calculator, find a suitable equation for the combined voltages, giving your answer in the form $V = V_0 \sin(at + b)$, where a, b and V_0 are constants, $a > 0$ and $0° \le b \le 180°$. [4]

Markscheme

recognizing that a is still 1 *A1*

$V_0 = 82.5$ *A1*

attempt to find an x-intercept of combined voltage   *(M1)*

$b = 26.2°$ $(26.1643\ldots°)$ **OR** any other correct x-intercept   *A1*

Note: May be seen in the final answer. Award *M1A0* for $b = 26$ with no working.

$$(V_{\text{TOT}} = 82.5 \sin(t + 26.2°) \ (82.5471\ldots \sin(t + 26.1643\ldots°)))$$

Note: Award at most *(M1)A1(A1)A0* if phase shift of $-153.835\ldots$ is seen in the final answer. In part (b), candidates may use $\arg(30e^{60i} + 60e^{10i})$ to determine the new phase shift, and hence could be awarded *M1* for this valid method.

[4 marks]

6. [Maximum mark: 8]

The time of sunrise, R hours after midnight, in Taipei can be modelled by

$$R = 1.08 \cos(0.0165t + 0.413) + 4.94,$$

where t is the day of the year 2021 (for example, $t = 2$ represents 2 January 2021).

The time of sunset, S hours after midnight, in Taipei can be modelled by

$$S = 1.15 \cos(0.0165t - 2.97) + 18.9.$$

The number of daylight hours, D, in Taipei during 2021 can be modelled by

$$D = a \cos(0.0165t + b) + c.$$

(a) Find the value of a, of b and of c. [6]

Markscheme

$$D = S - R$$

METHOD 1

$$= \mathrm{Re}\left(1.15\mathrm{e}^{(0.0165t - 2.97)\mathrm{i}}\right) - \mathrm{Re}\left(1.08\mathrm{e}^{(0.0165t + 0.413)\mathrm{i}}\right) \ (+18.9 - 4.94) \qquad \textbf{\textit{(M1)(A1)}}$$

$$= \mathrm{Re}\left(\mathrm{e}^{0.0165t\mathrm{i}}\left(1.15\mathrm{e}^{-2.97\mathrm{i}} - 1.08\mathrm{e}^{0.413\mathrm{i}}\right)\right) \ (+13.96) \qquad \textbf{\textit{(M1)}}$$

$$= \mathrm{Re}\left(\mathrm{e}^{0.0165t\mathrm{i}}\left(2.21379\ldots\mathrm{e}^{-2.85310\ldots\mathrm{i}}\right)\right) \ (+13.96) \qquad \textbf{\textit{(A1)}}$$

$$= 2.21\cos(0.0165t - 2.85) + 13.96 \quad (2.21379\ldots\cos(0.0165t - 2.85310\ldots) + 13.96)$$
      _A1A1_

$$(a = 2.21, \ b = -2.85, \ c = 13.96)$$

Note: Award **_A1_** for $2.21\cos(0.0165t - 2.85)$ and **_A1_** for "$+13.96$". The **_A1_** for 13.96 is independent of the previous marks.

METHOD 2

$$c = 13.9 - 4.94\ldots = 13.96 \qquad \textbf{\textit{A1}}$$

using a graph of D **_M1_**

maximum $(172.915\ldots, \ 16.1738\ldots)$ **_(A1)_**

minimum $(-17.4842\ldots, \ 11.7462\ldots)$ **_(A1)_**

EITHER

amplitude $16.1738\ldots - 11.7462\ldots = 4.4276\ldots$ **_A1_**

$$a = 2.21 \ (2.2138\ldots)$$

OR

$$a = 16.1738\ldots - 13.96 = 2.21 \ (2.2138\ldots) \qquad \textbf{\textit{A1}}$$

THEN

EITHER

when $t = 0, D = 11.8377\ldots$

$$11.8377\ldots = 2.2138\cos(b) + 13.96$$

$$b = -2.85 \ (2.85309\ldots) \qquad \textbf{\textit{A1}}$$

OR

$$b = -0.0165 \times 172.915\ldots = -2.85 \ (2.85309\ldots) \qquad \textbf{\textit{A1}}$$

[6 marks]

(b) Hence, or otherwise, find the largest number of daylight hours in Taipei during 2021 and the day of the year on which this occurs. [2]

Markscheme

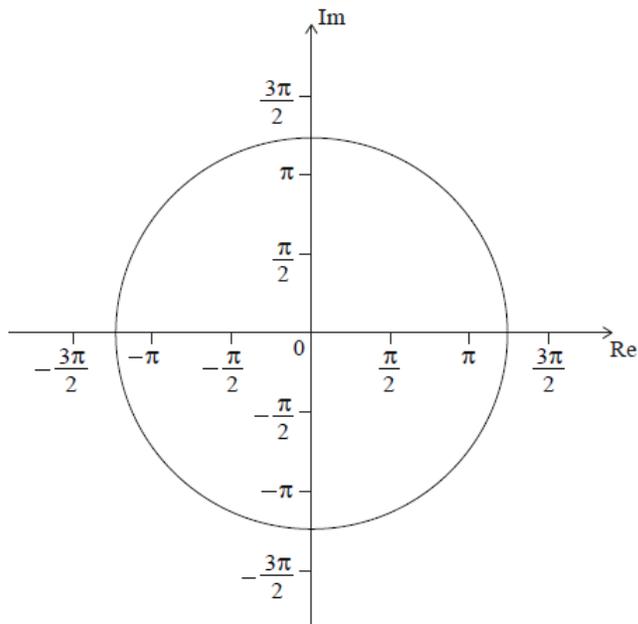$16.2 \ (16.1737\ldots)$ hours on day 173 **A1A1**

Note: Accept an answer of "day 172" for the second **A1**.

[2 marks]

7. [Maximum mark: 7] 22M.1.AHL.TZ1.10
The following Argand diagram shows a circle centre 0 with a radius of 4 units.

A set of points, $\{z_\theta\}$, on the Argand plane are defined by the equation

$z_\theta = \frac{1}{2}\theta e^{\theta i}$, where $\theta \geq 0$.

Plot on the Argand diagram the points corresponding to
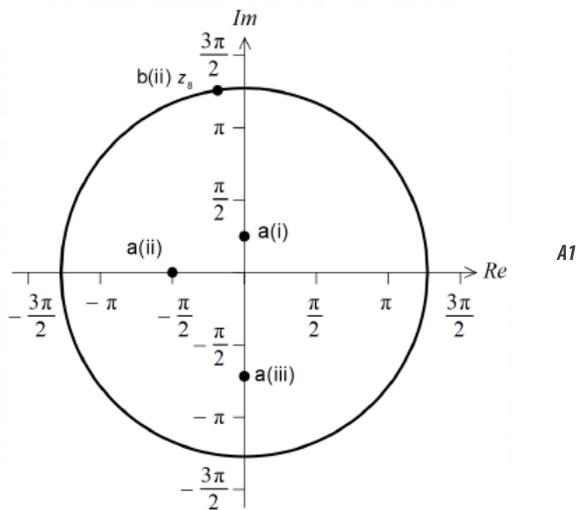
(a.i) $\theta = \frac{\pi}{2}$.

Markscheme

Note: Award *A1* for correct modulus and *A1* for correct argument for part (a)(i), and *A1* for other two points correct. The points may not be labelled, and they may be shown by line segments.

[1 mark]

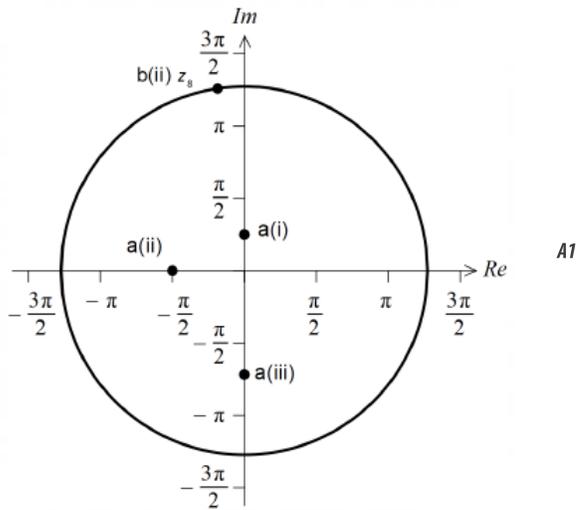(a.ii) $\theta = \pi$.   [1]

Markscheme

Note: Award *A1* for correct modulus and *A1* for correct argument for part (a)(i), and *A1* for other two points correct. The points may not be labelled, and they may be shown by line segments.

[1 mark]

(a.iii) $\theta = \frac{3\pi}{2}$. [1]

Note: Award *A1* for correct modulus and *A1* for correct argument for part (a)(i), and *A1* for other two points correct. The points may not be labelled, and they may be shown by line segments.

[1 mark]

Consider the case where $|z_\theta| = 4$.

(b.i)   Find this value of θ. [2]
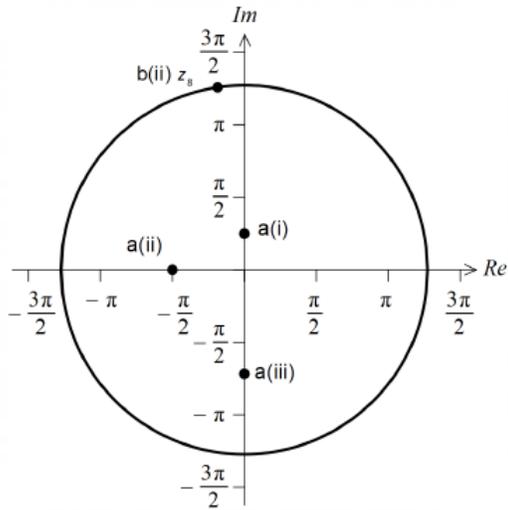
Markscheme

$\frac{1}{2}\theta = 4$ *(M1)*

$\Rightarrow \theta = 8$ *A1*

[2 marks]

(b.ii)   For this value of θ, plot the approximate position of z_θ on the Argand diagram. [2]

z_8 is shown in the diagram above   **A1A1**

Note: Award **A1** for a point plotted on the circle and **A1** for a point plotted in the second quadrant.

[2 marks]

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8.   [Maximum mark: 5]   22M.1.AHL.TZ2.13

An electric circuit has two power sources. The voltage, V_1, provided by the first power source, at time t, is modelled by

$$V_1 = \mathrm{Re}\left(2e^{3ti}\right).$$

The voltage, V_2, provided by the second power source is modelled by

$$V_2 = \mathrm{Re}\left(5e^{(3t+4)i}\right).$$

The total voltage in the circuit, V_T, is given by

$$V_T = V_1 + V_2.$$

(a)   Find an expression for V_T in the form $A\cos(Bt + C)$, where $A,\ B$ and C are real constants.   [4]

Markscheme

METHOD 1

recognizing that the real part is distributive   *(M1)*

$$V_T = \mathrm{Re}\left(2e^{3ti} + 5e^{3ti+4i}\right)$$

$= \text{Re}\left(e^{3ti}\left(2 + 5e^{4i}\right)\right)$ *(A1)*

(from the GDC) $2 + 5e^{4i} = 3.99088\ldots e^{-1.89418\ldots i}$ *(A1)*

Note: Accept arguments differing by 2π e.g. $4.38900\ldots$).

therefore $V_T = 3.99 \cos(3t - 1.89)$ $(3.99088\ldots \cos(3t - 1.89418\ldots))$ *A1*

Note: Award the last **A1** for the correct values of $A, \ B$ and C seen either in the required form or not. If method used is unclear and answer is partially incorrect, assume Method 2 and award appropriate marks eg. *(M1)A1A0A0* if only A value is correct.

METHOD 2

converting given expressions to cos form   *(M1)*

$V_T = 2 \cos 3t + 5 \cos(3t + 4)$

(from graph) $A = 3.99$ $(3.99088\ldots)$ *A1*

$V_T = 3.99 \cos(Bt + C)$

either by considering transformations or inserting points

$B = 3$ *A1*

$C = -1.89$ $(-1.89418\ldots)$ *A1*

Note: Accept arguments differing by 2π e.g. $4.38900\ldots$..

(so, $V_T = 3.99 \cos(3t - 1.89)$ $(3.99088\ldots \cos(3t - 1.89418\ldots))$)

Note: It is possible to have $A = 3.99, B = -3$ with $C = 1.89$ **OR** $A = -3.99, B = 3$ with $C = 1.25$ **OR** $A = -3.99, B = -3$ with $C = -1.25$ due to properties of the cosine curve.

[4 marks]

(b)   Hence write down the maximum voltage in the circuit.   [1]

Markscheme

maximum voltage is 3.99 $(3.99088\ldots)$ (units)   *A1*

[1 mark]

9. [Maximum mark: 13]
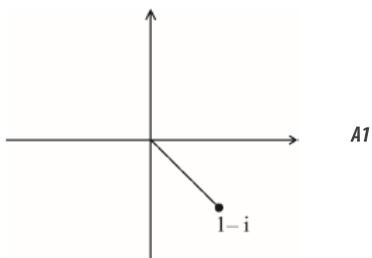
Let $z = 1 - \mathrm{i}$.

(a.i)   Plot the position of z on an Argand Diagram.   [1]

> **Markscheme**
>
>
>
> *[1 mark]*

(a.ii)   Express z in the form $z = a\mathrm{e}^{\mathrm{i}b}$, where $a,\ b \in \mathbb{R}$, giving the exact value of a and the exact value of b.   [2]

> **Markscheme**
>
> $z = \sqrt{2}\mathrm{e}^{\frac{\mathrm{i}\pi}{4}}$ *A1A1*
>
> **Note:** Accept an argument of $\dfrac{7\pi}{4}$. Do **NOT** accept answers that are not exact.
>
> *[2 marks]*

Let $w_1 = \mathrm{e}^{\mathrm{i}x}$ and $w_2 = \mathrm{e}^{\mathrm{i}\left(x - \frac{\pi}{2}\right)}$, where $x \in \mathbb{R}$.

(b.i)   Find $w_1 + w_2$ in the form $\mathrm{e}^{\mathrm{i}x}(c + \mathrm{i}d)$.   [2]

> **Markscheme**
>
> $$w_1 + w_2 = \mathrm{e}^{\mathrm{i}x} + \mathrm{e}^{\mathrm{i}\left(x - \frac{\pi}{2}\right)}$$
>
> $$= \mathrm{e}^{\mathrm{i}x}\left(1 + \mathrm{e}^{-\frac{\mathrm{i}\pi}{2}}\right) \quad \text{(M1)}$$
>
> $$= \mathrm{e}^{\mathrm{i}x}(1 - \mathrm{i}) \quad \text{A1}$$

(b.ii) Hence find $\mathrm{Re}(w_1 + w_2)$ in the form $A\,\cos(x - a)$, where $A > 0$ and $0 < a \leq \frac{\pi}{2}$. [4]

Markscheme

$$w_1 + w_2 = \mathrm{e}^{\mathrm{i}x} \times \sqrt{2}\mathrm{e}^{-\frac{\mathrm{i}\pi}{4}} \quad \textit{M1}$$

$$= \sqrt{2}\mathrm{e}^{\mathrm{i}\left(x - \frac{\pi}{4}\right)} \quad \textit{(A1)}$$

attempt extract real part using cis form   *(M1)*

$$\mathrm{Re}(w_1 + w_2) = \sqrt{2}\cos\left(x - \frac{\pi}{4}\right) \text{ OR } 1.4142\ldots\cos(x - 0.785398\ldots) \quad \textit{A1}$$

[4 marks]

The current, I, in an AC circuit can be modelled by the equation $I = a\,\cos(bt - c)$ where b is the frequency and c is the phase shift.

Two AC voltage sources of the same frequency are independently connected to the same circuit. If connected to the circuit alone they generate currents I_A and I_B. The maximum value and the phase shift of each current is shown in the following table.

Current	Maximum value	Phase shift
I_A	12 amps	0
I_B	12 amps	$\frac{\pi}{2}$

When the two voltage sources are connected to the circuit at the same time, the total current I_T can be expressed as $I_A + I_B$.

(c.i) Find the maximum value of I_T. [3]

Markscheme

$$I_t = 12\cos(bt) + 12\cos\left(bt - \frac{\pi}{2}\right) \quad \textit{(M1)}$$

$$I_t = 12\,\mathrm{Re}\left(\mathrm{e}^{\mathrm{i}bt} + \mathrm{e}^{\mathrm{i}\left(bt - \frac{\pi}{2}\right)}\right) \quad \textit{(M1)}$$

$$I_t = 12\sqrt{2}\cos\left(bt - \frac{\pi}{4}\right)$$

$$\text{max} = 12\sqrt{2}\ (= 17.0) \quad \textit{A1}$$

[3 marks]

(c.ii)   Find the phase shift of I_T. [1]

10.   [Maximum mark: 8]   21M.1.AHL.TZ1.9
Consider $w = \mathrm{i}z + 1$, where $w,\ z \in \mathbb{C}$.

Find w when

(a.i) $z = 2\mathrm{i}$. [2]

(a.ii) $z = 1 + \mathrm{i}$. [1]

Point z on the Argand diagram can be transformed to point w by two transformations.

(b)   Describe these two transformations and give the order in which they are applied. [3]

followed by a translation of $\begin{pmatrix} 1 \\ 0 \end{pmatrix}$ **A1**

OR

translation of $\begin{pmatrix} 0 \\ -1 \end{pmatrix}$ **A1**

followed by rotation of $90°$ (anticlockwise, centre at the origin) **A1A1**

Note: Award **A1** for "rotation" and A1 for "$90°$".

[3 marks]

(c) Hence, or otherwise, find the value of z when $w = 2 - \mathrm{i}$. [2]

Markscheme

EITHER

move 1 to left to $1 - \mathrm{i}$ *(M1)*

then rotate by $-90°$ to

$-1 - \mathrm{i}$ **A1**

OR

$\mathrm{i}z + 1 = 2 - \mathrm{i}$

$\mathrm{i}z = 1 - \mathrm{i}$

$z = \frac{1-\mathrm{i}}{\mathrm{i}}$ *(M1)*

$-1 - \mathrm{i}$ **A1**

[2 marks]

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11. [Maximum mark: 8]

It is given that $z_1 = 3 \operatorname{cis}\left(\frac{3\pi}{4}\right)$ and $z_2 = 2 \operatorname{cis}\left(\frac{n\pi}{16}\right)$, $n \in \mathbb{Z}^+$.

In parts (a)(i) and (a)(ii), give your answers in the form $r\mathrm{e}^{\mathrm{i}\theta}$, $r \geq 0$, $-\pi < \theta \leq \pi$.

(a.i) Find the value of $z_1{}^3$.

Markscheme

$$z_1{}^3 = 27\mathrm{e}^{\frac{\mathrm{i}\pi}{4}} \left(= 27\mathrm{e}^{0.785398\ldots\,\mathrm{i}}\right) \qquad \textbf{\textit{A1A1}}$$

Note: Award **A1** for 27 and **A1** for the angle in the correct form.

[2 marks]

(a.ii) Find the value of $\left(\frac{z_1}{z_2}\right)^4$ for $n = 2$.

[3]

Markscheme

$$\left(\frac{z_1}{z_2}\right)^4 = \left(\frac{81}{16}\right)\mathrm{e}^{\frac{\mathrm{i}\pi}{2}} \left(= 5.0625\mathrm{e}^{1.57079\ldots\,\mathrm{i}}\right) \qquad \textbf{\textit{A1A2}}$$

Note: Award **A1** for $\frac{81}{16}$, **A2** for the angle in the correct form and **A1** for the angle in incorrect form e.g. $\mathrm{cis}\frac{\pi}{2}$ and/or $\frac{5\pi}{2}$. Award **A1** if i is given in place of $\mathrm{cis}\frac{\pi}{2}$.

[3 marks]

(b) Find the least value of n such that $z_1 z_2 \in \mathbb{R}^+$.

[3]

Markscheme

$$z_1 z_2 = 6\,\mathrm{cis}\left(\frac{3\pi}{4} + \frac{n\pi}{16}\right) \qquad \textbf{\textit{(M1)}}$$

$$= 6\,\mathrm{cis}\left(\frac{12\pi + n\pi}{16}\right)$$

$$12\pi + n\pi = 32\pi \qquad \textbf{\textit{(M1)}}$$

$$n = 20 \qquad \textbf{\textit{A1}}$$

[3 marks]

12. [Maximum mark: 27]   21M.3.AHL.TZ1.1

A suitable site for the landing of a spacecraft on the planet Mars is identified at a point, A. The shortest time from sunrise to sunset at point A must be found.

Radians should be used throughout this question. All values given in the question should be treated as exact.
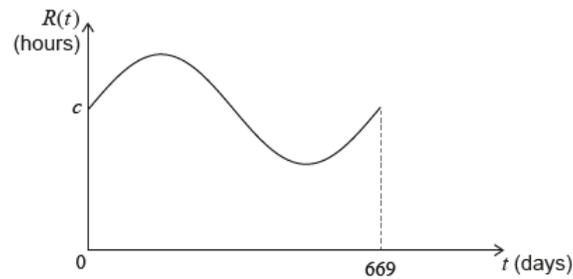
Mars completes a full orbit of the Sun in 669 Martian days, which is one Martian year.
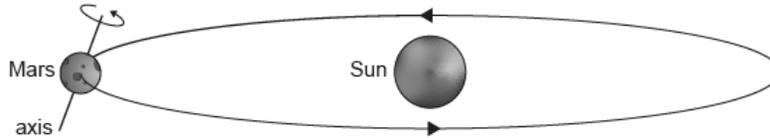
On day t, where $t \in \mathbb{Z}$, the length of time, in hours, from the start of the Martian day until sunrise at point A can be modelled by a function, $R(t)$, where

$$R(t) = a\,\sin(bt) + c, \ t \in \mathbb{R}.$$

The graph of R is shown for one Martian year.

(a)   Show that $b \approx 0.00939$.   [2]

> **Markscheme**
>
> recognition that period $= 669$ *(M1)*
>
> $b = \frac{2\pi}{669}$ **OR** $b = 0.00939190\ldots$ *A1*
>
> **Note:** Award *A1* for a correct expression leading to the given value or for a correct value of b to 4 sf or greater accuracy.
>
> $b \approx 0.00939$ *AG*
>
> *[2 marks]*

Mars completes a full rotation on its axis in 24 hours and 40 minutes.

(b)   Find the angle through which Mars rotates on its axis each hour.   [3]

> **Markscheme**
>
> length of day$= 24\frac{2}{3}$ hours   *(A1)*
>
> **Note:** Award *A1* for $\frac{2}{3}$, $0.666\ldots$, $0.\overline{6}$ or 0.667.
>
> $\frac{2\pi}{24\frac{2}{3}}$ *(M1)*

Note: Accept $\left(\frac{360}{24\frac{2}{3}}\right)$.

$= 0.255$ radians $\left(0.254723\ldots, \frac{3\pi}{37}, 14.5945\ldots^{\circ}\right)$ *A1*

[3 marks]

The time of sunrise on Mars depends on the angle, δ, at which it tilts towards the Sun. During a Martian year, δ varies from -0.440 to 0.440 radians.

The angle, ω, through which Mars rotates on its axis from the start of a Martian day to the moment of sunrise, at point A, is given by $\cos \omega = 0.839 \tan \delta, 0 \leq \omega \leq \pi$.

(c.i)   Show that the maximum value of $\omega = 1.98$, correct to three significant figures.   [3]

Markscheme

substitution of either value of δ into equation   *(M1)*

correct use of arccos to find a value for ω *(M1)*

Note: Both *(M1)* lines may be seen in either part (c)(i) or part (c)(ii).

$\cos \omega = 0.839 \tan(-0.440)$ *A1*

$\omega = 1.97684\ldots$

≈ 1.98 *AG*

Note: For substitution of 1.98 award *M0A0*.

[3 marks]

(c.ii)   Find the minimum value of ω.   [1]

Markscheme

$\delta = 0.440$

$\omega = 1.16 \; (1.16474\ldots)$ *A1*

[1 mark]

Use your answers to parts (b) and (c) to find

(d.i)   the maximum value of $R(t)$.   [2]

(d.ii)   the minimum value of $R(t)$.   [1]

(e)   Hence show that $a = 1.6$, correct to two significant figures.   [2]

(f)   Find the value of c.   [2]

OR

$c = 4.57258\ldots + 1.59408\ldots$ or $c = 7.76075\ldots - 1.59408\ldots$

THEN

$= 6.17 \ (6.16666\ldots)$ *A1*

Note: Accept 6.16 from use of rounded values. Follow through on their answers to part (d) and 1.6.

[2 marks]

Let $S(t)$ be the length of time, in hours, from the start of the Martian day until **sunset** at point A on day t. $S(t)$ can be modelled by the function

$S(t) = 1.5 \sin(0.00939t + 2.83) + 18.65.$

The length of time between sunrise and sunset at point A, $L(t)$, can be modelled by the function

$L(t) = 1.5 \sin(0.00939t + 2.83) - 1.6 \sin(0.00939t) + d.$

(g)   Find the value of d.   [2]

Markscheme

$d = 18.65 - 6.16666\ldots$ *(M1)*

$= 12.5 \ (12.4833\ldots)$ *A1*

Note: Follow through for 18.65 minus their answer to part (f).

[2 marks]

Let $f(t) = 1.5 \sin(0.00939t + 2.83) - 1.6 \sin(0.00939t)$ and hence $L(t) = f(t) + d$.

$f(t)$ can be written in the form $\text{Im}(z_1 - z_2)$, where z_1 and z_2 are complex functions of t.

(h.i)   Write down z_1 and z_2 in exponential form, with a constant modulus.   [3]

Markscheme

at least one expression in the form $re^{g(t)i}$ *(M1)*

$z_1 = 1.5e^{(0.00939t + 2.83)i}, \ z_2 = 1.6e^{(0.00939t)i}$ *A1A1*

[3 marks]

(h.ii)   Hence or otherwise find an equation for L in the form $L(t) = p\sin(qt + r) + d$, where
$p,\ q,\ r,\ d \in \mathbb{R}.$ [4]

Markscheme

EITHER

$z_1 - z_2 = 1.5\mathrm{e}^{(0.00939t+2.83)\mathrm{i}} - 1.6\mathrm{e}^{(0.00939t)\mathrm{i}}$

$= \mathrm{e}^{0.00939t\mathrm{i}}\left(1.5\mathrm{e}^{2.83\mathrm{i}} - 1.6\right)$ *(M1)*

$= \mathrm{e}^{0.00939t\mathrm{i}}\left(3.06249\ldots\mathrm{e}^{2.99086\ldots\mathrm{i}}\right)$ *(A1)(A1)*

OR

graph of L or f

$p = 3.06249\ldots$ *(A1)*

$r = -0.150729\ldots$ **OR** $r = 2.99086\ldots$ *(M1)(A1)*

Note: The p and r variables (or equivalent) must be seen.

THEN

$L(t) = 3.06\sin(0.00939t + 2.99) + 12.5$ *A1*

$(L(t) = 3.06248\ldots\sin(0.00939t + 2.99086\ldots) + 12.4833\ldots)$

Note: Accept equivalent forms, e.g. $L(t) = 3.06\sin(0.00939t - 0.151) + 12.5.$
Follow through on their answer to part (g) replacing 12.5.

[4 marks]

(h.iii)   Find, in hours, the shortest time from sunrise to sunset at point A that is predicted by this model.   [2]

Markscheme

shortest time between sunrise and sunset

$12.4833\ldots - 3.06249\ldots$ *(M1)*

$= 9.42$ hours $(9.420843\ldots)$ *A1*

Note: Accept 9.44 from use of 3 sf values.

[2 marks]