# Matrices revision [62 marks]

**1.** [Maximum mark: 4] EXM.1.AHL.TZ0.3

$A$ and $B$ are $2 \times 2$ matrices, where $A = \begin{bmatrix} 5 & 2 \\ 2 & 0 \end{bmatrix}$ and

$BA = \begin{bmatrix} 11 & 2 \\ 44 & 8 \end{bmatrix}$. Find $B$ [4]

Markscheme

* This question is from an exam for a previous syllabus, and may contain minor differences in marking or structure.

$B = (BA)A^{-1}$ ***(M1)***

$= -\frac{1}{4}\begin{pmatrix} 11 & 2 \\ 44 & 8 \end{pmatrix}\begin{pmatrix} 0 & -2 \\ -2 & 5 \end{pmatrix}$ ***(M1)***

$= -\frac{1}{4}\begin{pmatrix} -4 & -12 \\ -16 & -48 \end{pmatrix}$ ***(A1)***

$= \begin{pmatrix} 1 & 3 \\ 4 & 12 \end{pmatrix}$ ***(A1)***

**OR**

$\begin{pmatrix} a & b \\ c & d \end{pmatrix}\begin{pmatrix} 5 & 2 \\ 2 & 0 \end{pmatrix} = \begin{pmatrix} 11 & 2 \\ 44 & 8 \end{pmatrix}$ ***(M1)***

$\Rightarrow \left.\begin{array}{l} 5a + 2b = 11 \\ 2a \qquad\;\; = 2 \end{array}\right\}$

$\Rightarrow a = 1, b = 3$ ***(A1)***

$\left.\begin{array}{l} 5c + 2d = 44 \\ \quad 2c \;= 8 \end{array}\right\}$

$\Rightarrow c = 4, d = 12$ ***(A1)***

$B = \begin{pmatrix} 1 & 3 \\ 4 & 12 \end{pmatrix}$ ***(A1) (C4)***

**Note:** Correct solution with inversion (ie *AB* instead of *BA*) earns FT marks, (maximum ***[3 marks]***).

***[4 marks]***

**2.** [Maximum mark: 6] EXM.1.AHL.TZ0.28

Consider the matrix ***A*** $= \begin{pmatrix} \mathrm{e}^x & \mathrm{e}^{-x} \\ 2 + \mathrm{e}^x & 1 \end{pmatrix}$, where $x \in \mathbb{R}$.

Find the value of $x$ for which ***A*** is singular. [6]

Markscheme

finding det ***A*** $= \mathrm{e}^x - \mathrm{e}^{-x}(2 + \mathrm{e}^x)$ or equivalent ***A1***

***A*** is singular ⇒ det ***A*** = 0 ***(R1)***

$\mathrm{e}^x - \mathrm{e}^{-x}(2 + \mathrm{e}^x) = 0$

$\mathrm{e}^{2x} - \mathrm{e}^x - 2 = 0$ ***A1***

solving for $\mathrm{e}^x$ ***(M1)***

$\mathrm{e}^x > 0$ (or equivalent explanation) ***(R1)***

$\mathrm{e}^x = 2$

$x = \ln 2$ (only) ***A1 N0***

***[6 marks]***

**3.** [Maximum mark: 5] EXM.1.AHL.TZ0.32

If $\boldsymbol{A} = \begin{pmatrix} 1 & 2 \\ k & -1 \end{pmatrix}$ and $\boldsymbol{A}^2$ is a matrix whose entries are all 0, find $k$.

[5]

Markscheme

$\boldsymbol{A}^2 = \begin{pmatrix} 1 & 2 \\ k & -1 \end{pmatrix} \begin{pmatrix} 1 & 2 \\ k & -1 \end{pmatrix}$ ***M1***

$= \begin{pmatrix} 1+2k & 0 \\ 0 & 2k+1 \end{pmatrix}$ ***A2***

**Note:** Award ***A2*** for 4 correct, ***A1*** for 2 or 3 correct.

$1 + 2k = 0$ ***M1***

$k = -\frac{1}{2}$ ***A1***

***[5 marks]***

**4.** [Maximum mark: 6] EXM.1.AHL.TZ0.50

Given that $\boldsymbol{A} = \begin{pmatrix} 2 & 3 \\ 1 & -2 \end{pmatrix}$ and $\boldsymbol{B} = \begin{pmatrix} 2 & 0 \\ 0 & -3 \end{pmatrix}$, find $\boldsymbol{X}$ if $\boldsymbol{BX} = \boldsymbol{A} - \boldsymbol{AB}$.

[6]

Markscheme

**METHOD 1**

$\boldsymbol{A} - \boldsymbol{AB} = \begin{pmatrix} 2 & 3 \\ 1 & -2 \end{pmatrix} - \begin{pmatrix} 4 & -9 \\ 2 & 6 \end{pmatrix} = \begin{pmatrix} -2 & 12 \\ -1 & -8 \end{pmatrix}$ ***(M1)(A1)***

$\boldsymbol{X} = \boldsymbol{B}^{-1}(\boldsymbol{A} - \boldsymbol{AB}) = \boldsymbol{B}^{-1} \begin{pmatrix} -2 & 12 \\ -1 & -8 \end{pmatrix}$ ***(M1)***

$$= -\frac{1}{6}\begin{pmatrix} -3 & 0 \\ 0 & 2 \end{pmatrix}\begin{pmatrix} -2 & 12 \\ -1 & -8 \end{pmatrix}$$ ***(A1)***

$$= \begin{pmatrix} -1 & 6 \\ \frac{1}{3} & \frac{8}{3} \end{pmatrix}$$ ***(A2) (C6)***

**METHOD 2**

Attempting to set up a matrix equation ***(M2)***

$\boldsymbol{X} = \boldsymbol{B}^{-1}(\boldsymbol{A} - \boldsymbol{AB})$ ***(A2)***

$$= \begin{pmatrix} -1 & 6 \\ \frac{1}{3} & \frac{8}{3} \end{pmatrix}$$ (from GDC) ***(A2) (C6)***

***[6 marks]***

**5.** [Maximum mark: 6] EXM.1.AHL.TZ0.9

The matrix $\boldsymbol{A} = \begin{pmatrix} 1 & 2 & 0 \\ -3 & 1 & -1 \\ 2 & -2 & 1 \end{pmatrix}$ has inverse $\boldsymbol{A}^{-1} = \begin{pmatrix} -1 & -2 & -2 \\ 3 & 1 & 1 \\ a & 6 & b \end{pmatrix}$.

(a.i) Write down the value of $a$. [1]

Markscheme

* This question is from an exam for a previous syllabus, and may contain minor differences in marking or structure.

$a = 4$ ***A1 N1***

***[1 mark]***

(a.ii) Write down the value of $b$. [1]

Markscheme

$b = 7$ ***A1 N1***

***[1 mark]***

Consider the simultaneous equations

$$x + 2y = 7$$

$$-3x + y - z = 10$$

$$2x - 2y + z = -12$$

(b) Write these equations as a matrix equation. [1]

Markscheme

**EITHER**

$$A\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 7 \\ 10 \\ -12 \end{pmatrix}$$ ***A1 N1***

**OR**

$$\begin{pmatrix} 1 & 2 & 0 \\ -3 & 1 & -1 \\ 2 & -2 & 1 \end{pmatrix}\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 7 \\ 10 \\ -12 \end{pmatrix}$$ ***A1 N1***

***[1 mark]***

(c) Solve the matrix equation. [3]

Markscheme

$\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \boldsymbol{A}^{-1} \begin{pmatrix} 7 \\ 10 \\ -12 \end{pmatrix}$ (accept algebraic method) ***(M1)***

$\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} -3 \\ 5 \\ 4 \end{pmatrix}$ (accept $x = -3$, $y = 5$, $z = 4$) ***A2 N3***

***[3 marks]***

**6.** [Maximum mark: 6] EXM.1.AHL.TZ0.22

Let $\boldsymbol{A} = \begin{pmatrix} 1 & x & -1 \\ 3 & 1 & 4 \end{pmatrix}$ and $\boldsymbol{B} = \begin{pmatrix} 3 \\ x \\ 2 \end{pmatrix}$.

(a) Find ***AB***. [3]

Markscheme

* This question is from an exam for a previous syllabus, and may contain minor differences in marking or structure.

Attempting to multiply matrices ***(M1)***

$$\begin{pmatrix} 1 & x & -1 \\ 3 & 1 & 4 \end{pmatrix} \begin{pmatrix} 3 \\ x \\ 2 \end{pmatrix} = \begin{pmatrix} 3 + x^2 - 2 \\ 9 + x + 8 \end{pmatrix} \left( = \begin{pmatrix} 1 + x^2 \\ 17 + x \end{pmatrix} \right)$$

***A1A1 N3***

***[3 marks]***

(b) The matrix $\boldsymbol{C} = \begin{pmatrix} 20 \\ 28 \end{pmatrix}$ and $2\boldsymbol{AB} = \boldsymbol{C}$. Find the value of $x$. [3]

Markscheme

Setting up equation ***M1***

*eg* $2\begin{pmatrix} 1+x^2 \\ 17+x \end{pmatrix} = \begin{pmatrix} 20 \\ 28 \end{pmatrix}$, $\begin{pmatrix} 2+2x^2 \\ 34+2x \end{pmatrix} = \begin{pmatrix} 20 \\ 28 \end{pmatrix}$, $\begin{pmatrix} 1+x^2 \\ 17+x \end{pmatrix} = \begin{pmatrix} 10 \\ 14 \end{pmatrix}$

$$\begin{matrix} 2+2x^2 = 20 \\ 34+2x = 28 \end{matrix} \quad \begin{pmatrix} 1+x^2 = 10 \\ 17+x = 14 \end{pmatrix}$$ ***(A1)***

$x = -3$ ***A1 N2***

***[3 marks]***

**7.** [Maximum mark: 6] EXM.1.AHL.TZ0.52

(a) Find the inverse of the matrix $\begin{pmatrix} 1 & 2 & 1 \\ 1 & 1 & 2 \\ 2 & 1 & 4 \end{pmatrix}$. [2]

Markscheme

$$\begin{pmatrix} 1 & 2 & 1 \\ 1 & 1 & 2 \\ 2 & 1 & 4 \end{pmatrix}^{-1} = \begin{pmatrix} 2 & -7 & 3 \\ 0 & 2 & -1 \\ -1 & 3 & -1 \end{pmatrix}$$ ***A2 N2***

***[2 marks]***

(b) **Hence** solve the system of equations

$$x + 2y + z = 0$$

$$x + y + 2z = 7$$

$$2x + y + z = 17$$

[4]

Markscheme

In matrix form $A\mathbf{x} = B$ or $\mathbf{x} = A^{-1}B$ ***M1***

$x = 2, y = -3, z = 4$ ***A1A1A1 N0***

***[4 marks]***

**8.** [Maximum mark: 16] 24M.2.AHL.TZ2.5

The drivers of a delivery company can park their vans overnight either at its headquarters or at home.

Urvashi is a driver for the company. If Urvashi has parked her van overnight at headquarters on a given day, the probability that she parks her van at headquarters on the following day is $0.88$. If Urvashi has parked her van overnight at her home on a given day, the probability that she parks her van at home on the following day is $0.92$.

(a) Write down a transition matrix, $\boldsymbol{T}$, that shows the movement of Urvashi's van between headquarters and home. [2]

Markscheme

$$\begin{pmatrix} 0.88 & 0.08 \\ 0.12 & 0.92 \end{pmatrix}$$ ***M1A1***

**Note:** Award ***M1*** for correct values used, ***A1*** if in correct positions.

Accept alternative consistent matrix (e.g. the transpose or diagonal elements exchanged) and follow through to eigenvectors and initial state vector.

***[2 marks]***

On Monday **morning** she collected her van from headquarters where it was parked overnight.

(b) Find the probability that Urvashi's van will be parked at home at the end of the week on Friday **evening**. [3]

Markscheme

$5$ (seen) ***(A1)***

$$\begin{pmatrix} 0.88 & 0.08 \\ 0.12 & 0.92 \end{pmatrix}^5 \begin{pmatrix} 1 \\ 0 \end{pmatrix} = \begin{pmatrix} 0.596608 \\ 0.403392 \end{pmatrix}$$ **OR**

$$\begin{pmatrix} 0.88 & 0.8 \\ 0.12 & 0.92 \end{pmatrix}^5 = \begin{pmatrix} 0.596608 & 0.268928 \\ 0.403392 & 0.731072 \end{pmatrix}$$ ***(M1)***

$\mathrm{P}$(Friday evening)$= 0.403 \quad (0.403392)$ ***A1***

**Note:** Award ***A0M1A0*** for use of $4$ (and resulting probability $0.354$).

***[3 marks]***

(c) Write down the characteristic polynomial for the matrix $\boldsymbol{T}$. Give your answer in the form $\lambda^2 + b\lambda + c$. [2]

Markscheme

attempt to find $\det(\boldsymbol{A} - \lambda \boldsymbol{I})$ ***(M1)***

$$\begin{vmatrix} 0.88 - \lambda & 0.08 \\ 0.12 & 0.92 - \lambda \end{vmatrix}$$ ***OR***

$$(0.88 - \lambda)(0.92 - \lambda) - (0.12)(0.08)$$

$\lambda^2 - 1.8\lambda + 0.8$ ***A1***

***[2 marks]***

(d) Calculate eigenvectors for the matrix $\boldsymbol{T}$. [4]

Markscheme

eigenvalues are $0.8$ and $1$ ***(A1)***

**Note:** If no attempt is made to find eigenvectors, do not award ***A1*** for finding eigenvalues.

$$\begin{pmatrix} 0.88 & 0.08 \\ 0.12 & 0.92 \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} = 0.8 \begin{pmatrix} x \\ y \end{pmatrix}$$

$$0.88x + 0.08y = 0.8x$$

eigenvector=eg. $\begin{pmatrix} 1 \\ -1 \end{pmatrix}$ ***A1***

**EITHER**

$$\begin{pmatrix} 0.88 & 0.08 \\ 0.12 & 0.92 \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} = 1 \begin{pmatrix} x \\ y \end{pmatrix}$$ ***(M1)***

$$0.88x + 0.08y = x$$

$0.08y = 0.12x$

**OR**

eigenvalue $1$ gives

$$\begin{pmatrix} -0.12 & 0.08 \\ 0.12 & -0.08 \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$$ ***(M1)***

$-0.12x + 0.08y = 0$

$0.08y = 0.12x$

**Note:** Award ***M1*** for an attempt to find the eigenvector with eigenvalue $1$.

**THEN**

eigenvector = eg. $\begin{pmatrix} 2 \\ 3 \end{pmatrix}$ ***A1***

**Note:** Award ***A0A1M0A0*** if only $\begin{pmatrix} 1 \\ -1 \end{pmatrix}$ is seen and no eigenvalues are found.

***[4 marks]***

(e) Write down matrices $\boldsymbol{P}$ and $\boldsymbol{D}$ such that $\boldsymbol{T} = \boldsymbol{PDP}^{-1}$, where $\boldsymbol{D}$ is a diagonal matrix. [2]

Markscheme

$$\boldsymbol{D} = \begin{pmatrix} 1 & 0 \\ 0 & 0.8 \end{pmatrix},\ \boldsymbol{P} = \begin{pmatrix} 2 & 1 \\ 3 & -1 \end{pmatrix}$$ **OR**

$$\boldsymbol{D} = \begin{pmatrix} 0.8 & 0 \\ 0 & 1 \end{pmatrix},\ \boldsymbol{P} = \begin{pmatrix} 1 & 2 \\ -1 & 3 \end{pmatrix}$$ ***A1A1***

**Note:** Award ***A1*** for one of $\boldsymbol{P}$ or $\boldsymbol{D}$ correct. Do not award the second ***A1*** unless $\boldsymbol{P}$ and $\boldsymbol{D}$ are consistent.

***[2 marks]***

(f) Hence find the long-term probability that Urvashi's van is parked at home. [3]

Markscheme

**EITHER**

attempt to use $\boldsymbol{T}^n = \left(\boldsymbol{PDP}^{-1}\right)^n = \boldsymbol{PD}^n\boldsymbol{P}^{-1}$ ***M1***

**Note:** Award ***M1*** for their $\boldsymbol{D}^n$ seen.

limit of $\boldsymbol{D}^n$ calculated ***A1***

$$\begin{pmatrix} 2 & 1 \\ 3 & -1 \end{pmatrix}\begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}\begin{pmatrix} 2 & 1 \\ 3 & -1 \end{pmatrix}^{-1}$$

**Note:** $\begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}$ must be seen to award ***A1***.

**OR**

attempt to expand their $\boldsymbol{PD}^n\boldsymbol{P}^{-1}$ using explicit $\boldsymbol{P},\ \boldsymbol{P}^{-1}$ ***M1***

$$(\boldsymbol{T}^n =)\ \tfrac{1}{5}\begin{pmatrix}2 & 1\\ 3 & -1\end{pmatrix}\begin{pmatrix}1 & 0\\ 0 & 0.8^n\end{pmatrix}\begin{pmatrix}1 & 1\\ 3 & -2\end{pmatrix}$$

$$(\boldsymbol{T}^n =)\ \tfrac{1}{5}\begin{pmatrix}2+3(0.8^n) & 2-2(0.8^n)\\ 3-3(0.8^n) & 3+2(0.8^n)\end{pmatrix}$$ ***A1***

**Note:** Using this method, the limit of $0.8^n$ may be inferred and ***M1A1*** awarded.

**THEN**

$0.6$ ***A1***

**Note:** Multiplication by initial condition $\begin{pmatrix}1\\0\end{pmatrix}$ may be seen at any point as part of their method.

For an answer of $0.6$ from incomplete methods award a maximum of ***M1A0A0***, or if no working is seen, award ***M0A0A1***.

***[3 marks]***

**9.** [Maximum mark: 7] 23N.1.AHL.TZ0.15

The eating habits of students in a school are studied over a number of months. The focus of the study is whether non-vegetarians become vegetarians, and whether vegetarians remain vegetarians.

Each month, students choose between the vegetarian or non-vegetarian lunch options.

Once they have chosen for the month, they cannot change the option until the next month.

In any month during the study, it is noticed that the probability of a non-vegetarian becoming vegetarian the following month is $0.1$, and that the probability of a vegetarian remaining a vegetarian the following month is $0.8$.

This situation can be represented by the transition matrix

$$\boldsymbol{T} = \begin{pmatrix} 0.8 & 0.1 \\ 0.2 & 0.9 \end{pmatrix}.$$

(a) Interpret the value $0.9$ in $\boldsymbol{T}$ in terms of the changes in the eating habits of the students in the school. [1]

Markscheme

probability of non veg remaining non veg ***A1***

***[1 mark]***

(b) Find the eigenvalues of matrix $\boldsymbol{T}$. [3]

Markscheme

attempt to use $\det\,(\boldsymbol{A} - \lambda \boldsymbol{I}) = 0$ ***(M1)***

$$\begin{vmatrix} 0.8 - \lambda & 0.1 \\ 0.2 & 0.9 - \lambda \end{vmatrix} = 0$$

$(0.8 - \lambda)(0.9 - \lambda) - 0.1 \times 0.2 = 0$ ***(A1)***

$\lambda = 1;\ \lambda = 0.7$ ***A1***

***[3 marks]***

One of the eigenvectors of $\boldsymbol{T}$ is $\begin{pmatrix} -1 \\ 1 \end{pmatrix}$.

(c) Find another, non-parallel, eigenvector and interpret it in context. [3]

Markscheme

$-2a + b = 0$ ***M1***

$\boldsymbol{v}_1 = \begin{pmatrix} 1 \\ 2 \end{pmatrix}$ (accept any multiples of this answer) ***A1***

$\boldsymbol{v}_1$ means that in the long term the ratio of veg to non-veg is $1 : 2$
(in the long term one-third of students will be veg and two-thirds will not) ***A1***

***[3 marks]***