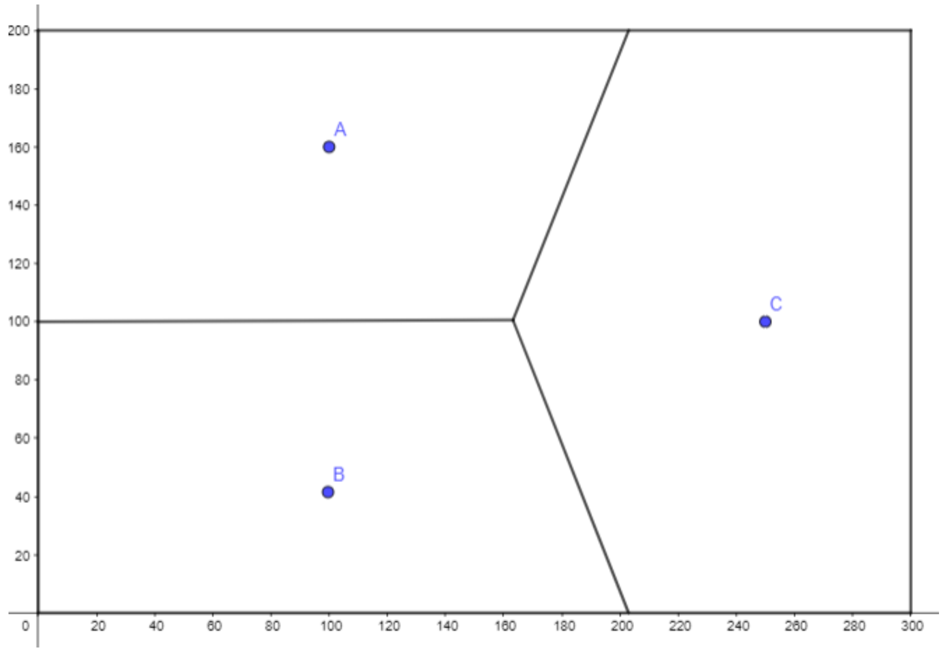
Mathematics: applications and interpretation

Practice paper 2 HL **Total 110**

1. [Maximum mark: 12]

The living accommodation on a university campus is in the shape of a rectangle with sides of length 200 metres and 300 metres.

There are three offices for the management of the accommodation set at the points A, B and C. These offices are responsible for all the students in the areas closest to the office. These areas are shown on the Voronoi diagram below. On this coordinate system the positions of A, B and C are (100, 160), (100, 40) and (250, 100) respectively.



The equation of the perpendicular bisector of [AC] is $5x-2y=615$.

(a) Find the x-coordinate of the point where this line intersects the line

(i) $y=200$

(ii) $y=100$ [3]

The manager of office C believes that he has more than one third of the area of the campus to manage.

(b) Find the area of campus managed by office C. [3]

(c) Hence or otherwise find the areas managed by offices A and B. [3]

(d) State a further assumption that must be made in order to use area covered as a measure of whether or not the manager of office C is responsible for more students than the managers of offices A and B. [1]

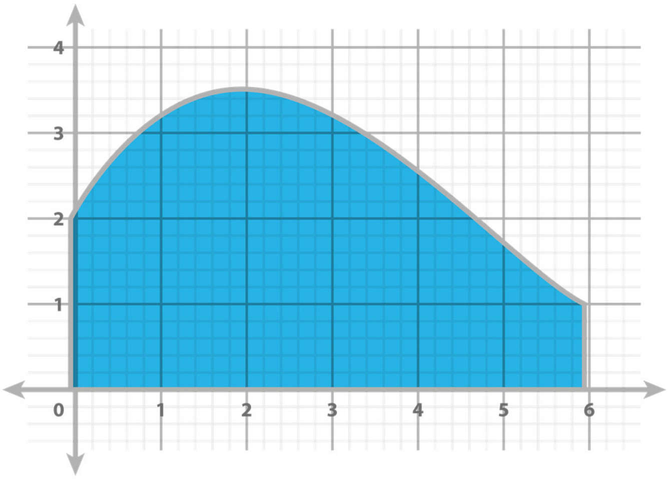
A new office is to be built within the triangle formed by A, B and C at a point as far as possible from the other three offices.

(e) Find the distance of this office from each of the other offices. [2]

2. [Maximum mark: 16]

A theatre set designer is designing a piece of flat scenery in the shape of a hill. The scenery is formed by a curve between two vertical edges of unequal height. One edge is 2 metres high and the other 1 metre high. The width of the scenery is 6 metres.

A coordinate system is formed with the origin at the foot of the 2 metres high edge. In this coordinate system the highest point of the piece of scenery is at $(2, 3.5)$.



A set designer wishes to work out an approximate value for the area of the scenery $(A\ \text{m}^2)$.

(a) Explain why $A < 21$. [1]

(b) By dividing the area between the curve and the x-axis into two trapezoids of unequal width show that $A > 14.5$, justifying the direction of the inequality. [4]

In order to obtain a more accurate measure for the area the designer decides to model the curved edge with the polynomial $h(x) = ax^3 + bx^2 + cx + d \quad a,b,c,d \in \mathbb{R}$ where h metres is the height of the curved edge a horizontal distance x meters from the origin.

(c) Write down the value of d. [1]

(d) Use differentiation to show that $12a + 4b + c = 0$. [2]

(e) Determine two other linear equations in a, b and c. [3]

(f) Hence find an expression for $h(x)$. [3]

(g) Use the expression found in (f) to calculate a value for A. [2]

3. [Maximum mark: 11]

Dana has collected some data regarding the heights *h* (metres) of waves against a pier at 50 randomly chosen times in a single day. This data is shown in the table below.

Height (*h*)	0≤*h* <0.5	0.5≤*h* <0.6	0.6≤*h* <0.7	0.7≤*h* <0.8	0.8≤*h* <0.9	0.9≤*h* <1.0	1.0≤*h* <1.1	1.1≤*h* <1.2	h≥1.2
Frequency	5	2	4	9	10	7	7	3	3

She wishes to perform a χ^2-test at the 5% significance level to see if the height of waves could be modelled by a normal distribution. Her null hypothesis is

H_0: The data can be modelled by a normal distribution.

From the table she calculates the mean of the heights in her sample to be 0.828 m and the standard deviation of the heights s_n to be 0.257 m.

(a) Use the given value of s_n to find the value of s_{n-1}. [2]

She calculates the expected values for each interval under this null hypothesis, and some of these values are shown in the table below.

Height (h)	0≤h <0.5	0.5≤h <0.6	0.6≤h <0.7	0.7≤h <0.8	0.8≤h <0.9	0.9≤h <1.0	1.0≤h <1.1	1.1≤h <1.2	h≥1.2
Frequency	5.2	4.3	6.1	*a*	*b*	6.8	5.3	3.6	3.8

(b) Find the value of *a* and the value of *b,* giving your answers correct to one decimal place. [3]

(c) Find the value of the χ^2 test statistic $\left(\chi^2_{calc}\right)$ for this test. [2]

(d) Determine the degrees of freedom for Dana's test. [2]

It is given that the critical value for this test is 9.49.

(e) State the conclusion of the test in context. Use your answer to part (c) to justify your conclusion. [2]

4. [Maximum mark: 16]

Jorge is carefully observing the rise in sales of a new app he has created.

The number of sales in the first four months is shown in the table below.

Month (*t*)	1	2	3	4
Sales (*N*)	40	52	70	98

Jorge believes that the increase is exponential and proposes to model the number of sales *N* in month *t* with the equation

$$N = A\mathrm{e}^{rt}, \; A, r \in \mathbb{R}$$

(a) Show that Jorge's model satisfies the differential equation

$$\frac{\mathrm{d}N}{\mathrm{d}t} = rN$$ [2]

Jorge plans to adapt Euler's method to find an approximate value for *r*.

With a step length of one month the solution to the differential equation can be approximated using Euler's method where

$$N(n+1) \approx N(n) + 1 \times N'(n), \; n \in \mathbb{N}$$

(b) Show that $r \approx \dfrac{N(n+1) - N(n)}{N(n)}$ [3]

(c) Hence find three approximations for the value of *r*. [3]

Jorge decides to take the mean of these values as the approximation of *r* for his model. He also decides the graph of the model should pass through the point (2, 52).

(d) Find the equation for Jorge's model. [3]

(e) Find the sum of the square residuals for Jorge's model using the values $t = 1,\ 2,\ 3,\ 4$. [2]

The sum of the square residuals for these points for the least squares regression model is approximately 6.555.

(f) (i) Comment how well Jorge's model fits the data.

(ii) Give two possible sources of error in the construction of his model. [3]

5. [Maximum mark: 19]

A change in grazing habits has resulted in two species of herbivore, X and Y, competing for food on the same grasslands. At time t=0 environmentalists begin to record the sizes of both populations. Let the size of the population of X be x, and the size of the population Y be y. The following model is proposed for predicting the change in the sizes of the two populations:

$$\dot{x} = 0.3x - 0.1y$$
$$\dot{y} = -0.2x + 0.4y$$
$$\text{for } x, y > 0$$

(a) For this system of coupled differential equations find

(i) the eigenvalues
(ii) the eigenvectors. [6]

(b) Hence write down the general solution of the system of equations. [1]

(c) Sketch the phase portrait for this system, for x, $y > 0$. [3]

On your sketch show

- the equation of the line defined by the eigenvector in the first quadrant
- at least two trajectories either side of this line using arrows on those trajectories to represent the change in populations as t increases

When $t = 0$ X has a population of 2000.

(d) Write down a condition on the size of the initial population of Y if it is to avoid its population reducing to zero. [1]

(e) It is known that Y has an initial population of 2900.

(i) Find the value of t at which $x = 0$.

(ii) Find the population of Y at this value of t. Give your answer to the nearest 10 herbivores. [8]

6. [Maximum mark: 18]

The masses in kilograms of melons produced by a farm can be modelled by a normal distribution with a mean of 2.6 kg and a standard deviation of 0.5 kg.

(a) Find the probability that a melon selected at random will have a mass greater than 3.0 kg. [2]

(b) Find the probability that two melons picked at random and independently of each other will

(i) both have a mass greater than 3.0 kg

(ii) have a total mass greater than 6.0 kg. [6]

One year due to favourable weather conditions it is thought that the mean mass of the melons has increased.

The owner of the farm decides to take a random sample of 16 melons to test this hypothesis at the 5% significance level, assuming the standard deviation of the masses of the melons has not changed.

(c) Write down the null and alternative hypotheses for the test. [1]

(d) Find the critical region for this test. [4]

Unknown to the farmer the favourable weather conditions have led to all the melons having 10% greater mass than the model described above.

(e) Find the mean and standard deviation of the mass of the melons for this year. [3]

(f) Find the probability of a Type II error in the owner's test. [2]

7. [Maximum mark: 18]

A ball is attached to the end of a string and spun horizontally. Its position relative to a given point, O, at time *t* seconds, $t \geq 0$, is given by the equation

$$\boldsymbol{r} = \begin{pmatrix} 1.5\cos\left(0.1t^2\right) \\ 1.5\sin\left(0.1t^2\right) \end{pmatrix}$$ where all displacements are in metres.

(a) Show that the ball is moving in a circle with its centre at O and state the radius of the circle. [4]

(b) (i) Find an expression for the velocity of the ball at time *t*.

(ii) Hence show that the velocity of the ball is always perpendicular to the position vector of the ball. [4]

(c) (i) Find an expression for the acceleration of the ball at time *t*.

The string breaks when the magnitude of the ball's acceleration exceeds 20 ms^{-2}.

(ii) Find the value of *t* at the instant the string breaks.

(iii) How many complete revolutions has the ball completed from $t = 0$ to the instant at which the string breaks? [10]