Functions, exponents and logarithms (GDC) [44 marks]

1. [Maximum mark: 17] EXN.2.SL.TZ0.9

The temperature $T\ ^\circ\mathrm{C}$ of water t minutes after being poured into a cup can be modelled by $T = T_0\mathrm{e}^{-kt}$ where $t \geq 0$ and $T_0,\ k$ are positive constants.

The water is initially boiling at $100\ ^\circ\mathrm{C}$. When $t = 10$, the temperature of the water is $70\ ^\circ\mathrm{C}$.

(a) Show that $T_0 = 100$. [1]

(b) Show that $k = \frac{1}{10}\ln\frac{10}{7}$. [3]

(c) Find the temperature of the water when $t = 15$. [2]

(d) Sketch the graph of T versus t, clearly indicating any asymptotes with their equations and stating the coordinates of any points of intersection with the axes. [4]

(e) Find the time taken for the water to have a temperature of $50\ ^\circ\mathrm{C}$. Give your answer correct to the nearest second. [4]

(f) The model for the temperature of the water can also be expressed in the form $T = T_0a^{\frac{t}{10}}$ for $t \geq 0$ and a is a positive constant.

Find the exact value of a. [3]

2. [Maximum mark: 6] 24M.2.SL.TZ2.4

The loudness of a sound, L, measured in decibels, is related to its intensity, I units, by $L = 10\log_{10}\left(I \times 10^{12}\right)$.

Consider two sounds, S_1 and S_2.

S_1 has an intensity of 10^{-6} units and a loudness of 60 decibels.

S_2 has an intensity that is twice that of S_1.

(a) State the intensity of S_2. [1]

(b) Determine the loudness of S_2. [2]

The maximum loudness of thunder in a thunderstorm was measured to be 115 decibels.

(c) Find the corresponding intensity, I, of the thunder. [3]

3. [Maximum mark: 7] 23M.2.SL.TZ1.7

The temperature of a cup of tea, t minutes after it is poured, can be modelled by $H(t) = 21 + 75\mathrm{e}^{-0.08t},\ t \geq 0$. The temperature is measured in degrees Celsius ($^\circ\mathrm{C}$).

(a.i) Find the initial temperature of the tea. [1]

(a.ii) Find the temperature of the tea three minutes after it is poured. [1]

(b) After k minutes, the tea will be below $67\ ^\circ\mathrm{C}$ and cool enough to drink.

Find the least possible value of k, where $k \in \mathbb{Z}^+$. [3]

As the tea cools, $H(t)$ approaches the temperature of the room, which is constant.

(c) Find the temperature of the room. [2]

4. [Maximum mark: 7] 22N.2.SL.TZ0.5

The population of a town t years after 1 January 2014 can be modelled by the function

$P(t) = 15\,000\mathrm{e}^{kt}$, where $k < 0$ and $t \geq 0$.

It is known that between 1 January 2014 and 1 January 2022 the population decreased by 11%.

Use this model to estimate the population of this town on 1 January 2041. [7]

5. [Maximum mark: 7] 21M.2.SL.TZ2.6

All living plants contain an isotope of carbon called carbon-14. When a plant dies, the isotope decays so that the amount of carbon-14 present in the remains of the plant decreases. The time since the death of a plant can be determined by measuring the amount of carbon-14 still present in the remains.

The amount, A, of carbon-14 present in a plant t years after its death can be modelled by $A = A_0\mathrm{e}^{-kt}$ where $t \geq 0$ and $A_0,\ k$ are positive constants.

At the time of death, a plant is defined to have 100 units of carbon-14.

(a) Show that $A_0 = 100$. [1]

The time taken for half the original amount of carbon-14 to decay is known to be 5730 years.

(b) Show that $k = \frac{\ln 2}{5730}$. [3]

(c) Find, correct to the nearest 10 years, the time taken after the plant's death for 25% of the carbon-14 to decay. [3]