7D Arithmetic series

When you add up terms of an arithmetic sequence you get an arithmetic series. There is a formula for the sum of an arithmetic series, the proof of which is not required in the IB. See Fill-in proof 4 'Arithmetic series and the story of Gauss' on the CD-ROM if you are interested.

There are two different forms for the formula.

KEY POINT 7.4

If you know the first and last terms:

$$S_n = \frac{n}{2}(u_1 + u_n)$$

If you know the first term and the common difference:

$$S_n = \frac{n}{2}\left(2u_1 + (n-1)d\right)$$

Worked example 7.9

Find the sum of the first 30 terms of an arithmetic progression with first term 8 and common difference 0.5.

We have all the information we need to use the second formula

$$S_{30} = \frac{30}{2}(2 \times 8 + (30-1) \times 0.5) = 457.5$$

Sometimes you have to interpret the question carefully to be sure that it is about an arithmetic sequence.

Worked example 7.10

Find the sum of all the multiples of 3 between 100 and 1000.

Write out the first few terms to see what is happening

Sum $= 102 + 105 + 108 + \ldots + 999$

This is an arithmetic series with $u_1 = 102$ and $d = 3$

To use either sum formula, we also need to know how many terms are in this sequence

We do this by setting $u_n = 999$

$$999 = 102 + 3(n-1)$$
$$\Leftrightarrow 897 = 3(n-1)$$
$$\Leftrightarrow \quad n = 300$$

Use the first sum formula

$$S_{300} = \frac{300}{2}(102 + 999) = 165\,150$$

You must be able to work backwards too; given information which includes the sum of the series, you may be asked to find out how many terms are in the series. Remember that the number of terms can only be a positive integer.

Worked example 7.11

An arithmetic sequence has first term 5 and common difference 10.

If the sum of all the terms is 720, how many terms are in the sequence?

We need to find n and it is the only unknown in the second sum formula

$$720 = \frac{n}{2}(2\times 5 + (n-1)\times 10)$$
$$= \frac{n}{2}(10 + 10n - 10)$$
$$= 5n^2$$

Solve this equation

$$n^2 = 144$$
$$n = \pm 12$$

But n must be a positive integer, so $n = 12$

Exercise 7D

1. Find the sum of the following arithmetic sequences:
   (a) (i) 12, 33, 54, … (17 terms)
   (ii) −100, −85, −70, … (23 terms)
   (b) (i) 3, 15, …, 459
   (ii) 2, 11, …, 650
   (c) (i) 28, 23, …, −52
   (ii) 100, 97, …,40
   (d) (i) 15, 15.5, …, 29.5
   (ii) $\frac{1}{12}, \frac{1}{6}, \ldots, 1.5$

2. An arithmetic sequence has first term 4 and common difference 8.

   How many terms are required to get a sum of:
   (a) (i) 676 (ii) 4096 (iii) 11236
   (b) $x^2, x > 0$

3. The second term of an arithmetic sequence is 7. The sum of the first four terms of the sequence is 12. Find the first term, a, and the common difference, d, of the sequence. *[5 marks]*

4. Consider the arithmetic series $2 + 5 + 8 + \ldots$:

(a) Find an expression for S_n, the sum of the first n terms.

(b) Find the value of n for which $S_n = 1365$. *[5 marks]*

5. Find the sum of the positive terms of the arithmetic sequence 85, 78, 71, ... *[6 marks]*

6. The second term of an arithmetic sequence is 6. The sum of the first four terms of the arithmetic sequence is 8. Find the first term, a, and the common difference, d, of the sequence. *[6 marks]*

7. Consider the arithmetic series $-6 + 1 + 8 + 15 + \ldots$

Find the least number of terms so that the sum of the series is greater than 10 000. *[6 marks]*

8. The sum of the first n terms of an arithmetic sequence is $S_n = 3n^2 - 2n$. Find the nth term u_n. *[6 marks]*

9. A circular disc is cut into twelve sectors whose angles are in an arithmetic sequence.

The angle of the largest sector is twice the angle of the smallest sector. Find the size of the angle of the smallest sector. *[6 marks]*

10. The ratio of the fifth term to the twelfth term of a sequence in an arithmetic progression is $\frac{6}{13}$.

If each term of this sequence is positive, and the product of the first term and the third term is 32, find the sum of the first 100 terms of this sequence. *[7 marks]*

11. What is the sum of all three-digit numbers which are multiples of 14 but not 21? *[8 marks]*

7E Geometric sequences

Geometric sequences have a constant ratio, called the *common ratio, r*, between terms:

$$u_{n+1} = r \times u_n$$

So examples of geometric sequences might be:

$1, 2, 4, 8, 16, \ldots \ (r = 2)$

$100, 50, 25, 12.5, 6.25, \ldots \ (r = \frac{1}{2})$

$1, -3, 9, -27, 81, \ldots \ (r = -3)$

As with arithmetic sequences, we also need to know the first term to fully define a geometric sequence. Again this is normally given the symbol u_1.

To get immediately to the deductive rule, we can see that to get to the nth term you start at the first term and multiply by the common ratio $n - 1$ times.

KEY POINT 7.5

$$u_n = u_1 r^{n-1}$$

Worked example 7.12

The seventh term of a geometric sequence is 13. The ninth term is 52.

What values could the common ratio take?

Write an expression for the seventh term in terms of u_1 and r	$u_7 = u_1 r^6$
But $u_7 = 13$	$13 = u_1 r^6 \quad (1)$
Repeat the same process for the ninth term	$52 = u_1 r^8 \quad (2)$
Solve the two equations simultaneously. Divide to eliminate u_1	$(2) \div (1)$ $4 = r^2$ $\Leftrightarrow r = \pm 2$

EXAM HINT

Notice that the question asked for values rather than value. This is a big hint that there is more than one solution.

When questions on geometric sequences ask what term satisfies a particular condition, you will usually use logarithms to solve an equation. Be careful when dealing with logarithms and inequalities; if you divide by the logarithm of a number less than 1 then you must flip the inequality.

Worked example 7.13

A geometric sequence has first term 10 and common ratio $\frac{1}{3}$. What is the first term that is less than 10^{-6}?

Express the condition as an inequality	$10 \times \left(\frac{1}{3}\right)^{n-1} < 10^{-6}$
	$\Leftrightarrow \left(\frac{1}{3}\right)^{n-1} < 10^{-7}$
The unknown is in the power so we solve it using logarithms	$\Leftrightarrow \log\left(\frac{1}{3}\right)^{n-1} < \log 10^{-7}$

continued . . .

Use the logarithm law $\log_a x^p = p\log_a x$

$\log\left(\frac{1}{3}\right) < 0$ so reverse the inequality when dividing

$$\Leftrightarrow \quad (n-1)\log\left(\frac{1}{3}\right) < \log 10^{-7}$$

$$\Leftrightarrow \quad (n-1) > \frac{\log 10^{-7}}{\log\left(\frac{1}{3}\right)}$$

$$\Leftrightarrow \quad n > 15.7 \text{ (3SF)}$$

But n is a whole number so the first term satisfying the condition is 16

A particular problem is when the common ratio is negative, as we cannot take the log of a negative number. However, we can get around this problem using the fact that a negative number raised to an even power is always positive.

Inequalities are covered in Prior learning Section L.

Worked example 7.14

A geometric sequence has first term 2 and common ratio –3. What term has the value –4374?

Write the information given as an equation

$$-4374 = 2 \times (-3)^{n-1}$$

Multiply both sides by –3 to make the left hand side positive

$$\Leftrightarrow 13\,122 = 2 \times (-3)^n$$

Since the LHS is positive the RHS must also be positive, so n must be even and we can replace $(-3)^n$ with $(3)^n$

Since both sides must be positive:

$$13\,122 = 2 \times (3)^n$$

Solve the equation using logarithms

$$6561 = 3^n$$

$$\log 6561 = n \log 3$$

$$n = \frac{\log 6561}{\log 3} = 8$$

In Worked example 7.14, you might have tried taking logarithms at the first line. Although this would have meant logs of negative numbers, using the rules of logs leads to the correct answer. This suggests that there may be some interpretation of logs of negative numbers, and when we meet complex numbers there will indeed be an interpretation. So does the logarithm of a negative number 'exist'? To some extent in mathematics if a concept is useful, that is enough to justify its introduction.

Exercise 7E

1. Find an expression for the nth term of these geometric sequences:
   - (a) (i) 6, 12, 24, … (ii) 12, 18, 27, …
   - (b) (i) 20, 5, 1.25, … (ii) $1, \frac{1}{2}, \frac{1}{4}, \ldots$
   - (c) (i) 1, −2, 4, … (ii) 5, −5, 5, …
   - (d) (i) $a, ax, ax^2, \ldots$ (ii) $3, 6x, 12x^2, \ldots$

2. How many terms are there in the following geometric sequences?
   - (a) (i) 6, 12, 24, …, 24576 (ii) 20, 50, …, 4882.8125
   - (b) (i) $1, -3, \ldots, -19683$ (ii) 2, −4, 8, …, −1024
   - (c) (i) $\frac{1}{2}, \frac{1}{4}, \ldots, \frac{1}{1024}$ (ii) $3, 2, \frac{4}{3}, \ldots, \frac{128}{729}$

3. How many terms are needed in the following geometric sequences to get within 10^{-9} of zero?
   - (a) (i) $5, 1, \frac{1}{5}, \ldots$ (ii) 0.6, 0.3, 0.15, …
   - (b) (i) 4, −2, 1, … (ii) −125, 25, −5, …

4. The second term of a geometric sequence is 6 and the fifth term is 162. Find the tenth term. *[5 marks]*

5. The third term of a geometric sequence is 112 and the sixth term is 7168.
   Which term takes the value 1 835 008? *[5 marks]*

6. Which is the first term of the sequence $\frac{2}{5}, \frac{4}{25}, \ldots, \frac{2^n}{5^n}$ that is less than 10^{-6}? *[6 marks]*

7. The difference between the fourth and the third term of a geometric sequence is $\frac{75}{8}$ times the first term. Find all possible values of the common ratio. *[6 marks]*

8. The third term of a geometric progression is 12 and the fifth term is 48. Find the two possible values of the eighth term. *[6 marks]*

9. The first three terms of a geometric sequence are a, $a + 14$ and $9a$. Find all possible values of a. *[6 marks]*

10. The three terms a, 1, b are in arithmetic progression.
The three terms 1, a, b are in geometric progression.
Find the value of a and b given that $a \neq b$. *[7 marks]*

11. The sum of the first n terms of an arithmetic sequence $\{u_n\}$ is given by the formula $S_n = 4n^2 - 2n$. Three terms of this sequence, u_2, u_m and u_{32}, are consecutive terms in a geometric sequence. Find m. *[7 marks]*

7F Geometric series

As with arithmetic series there is a complicated formula for the sum of geometric sequences. See Fill-in proof 5 'Self-similarity and geometric series', on the CD-ROM for the derivation.

KEY POINT 7.6

$$S_n = \frac{u_1(1-r^n)}{1-r} \qquad (r \neq 1)$$

or equivalently

$$S_n = \frac{u_1(r^n-1)}{r-1} \qquad (r \neq 1)$$

We generally use the first of these formulae when the common ratio is less than one and the second when the common ratio is greater than one. This avoids working with negative numbers.

Worked example 7.15

Find the exact value of the sum of the first 6 terms of the geometric sequence with first term 8 and common ratio $\frac{1}{2}$.

$r < 1$, so use the first sum formula

$$S_6 = \frac{8\left(1-\left(\frac{1}{2}\right)^6\right)}{1-\frac{1}{2}}$$

$$= \frac{8\left(1-\frac{1}{64}\right)}{\frac{1}{2}}$$

$$= 16\left(\frac{63}{64}\right)$$

$$= \frac{63}{4}$$

We may be given information about the sum and have to deduce other information.

Worked example 7.16

How many terms are needed for the sum of the geometric series $3 + 6 + 12 + 24 + \ldots$ to exceed 10 000?

State the values of u_1 and r

$u_1 = 3$
$r = 2$

$r > 1$, so use the second sum formula and express the condition as an inequality

$$S_n = \frac{3(2^n - 1)}{2 - 1} > 10\,000$$

The unknown n is in the power, so use logarithms to solve the inequality

$$3(2^n - 1) > 10\,000$$

$$2^n > \frac{10\,003}{3}$$

$$\Leftrightarrow \log 2^n > \log\left(\frac{10\,003}{3}\right)$$

$$\Leftrightarrow n\log 2 > \log\left(\frac{10\,003}{3}\right)$$

$$\Leftrightarrow \quad n > \log\left(\frac{10\,003}{3}\right) \div \log 2$$

$$n > 11.7 \text{ (3SF)}$$

But n must be a whole number so 12 terms are needed.

Exercise 7F

Geometric series get very large very quickly. A mathematical legend involving the supposed inventor of chess, Sissa Ibn Dahir, illustrates how poor our intuition is with large numbers. It is explored on Supplementary sheet 6 'The chess legend and extreme numbers'.

1. Find the sums of the following geometric series. (There may be more than one possible answer!)

(a) (i) 7, 35, 175, … (10 terms)

(ii) 1152, 576, 288, … (12 terms)

(b) (i) 16, 24, 36, …, 182.25

(ii) 1, 1.1, 1.21, …, 1.771651

(c) (i) First term 8, common ratio −3, last term 52 488

(ii) First term −6, common ratio −3, last term 13 122

(d) (i) Third term 24, fifth term 6, 12 terms

(ii) Ninth term 50, thirteenth term 0.08, last term 0.0032

2. Find the possible values of the common ratio if the:

(a) (i) first term is 11 and the sum of the first 12 terms is 2 922 920

(ii) first term is 1 and the sum of the first 6 terms is 1.249 94

(b) (i) first term 12 and the sum of the first 6 terms is −79 980

(ii) first term is 10 and the sum of the first 4 terms is 1.

3. The nth term, u_n, of a geometric sequence is given by $u_n = 3 \times 5^{n+2}, n \in \mathbb{Z}^+$.

(a) Find the common ratio, r.

(b) Hence or otherwise, find S_n, the sum of the first n terms of this sequence. *[5 marks]*

4. The sum of the first three terms of a geometric sequence is $23\frac{3}{4}$ and the sum of the first four terms is $40\frac{5}{8}$. Find the first term and the common ratio. *[6 marks]*

5. (a) A geometric sequence has first term 1 and common ratio x. Express the sum of the first four terms as a polynomial in x.

(b) Factorise $x^6 - 1$ into a linear factor and a polynomial of order 5. *[6 marks]*

7G Infinite geometric series

If we keep adding together terms of any arithmetic sequence the answer grows (or decreases) without limit. The series is **divergent**.

Sometimes this happens with geometric series too, but there are cases where the sum gets closer and closer to a finite number. The series is **convergent**.

Not all geometric series converge. To decide which ones do, we need to use the formula for a geometric series from Key point 7.6.

$$S_n = \frac{u_1(1-r^n)}{1-r}$$

and look at the r^n term. We are interested in what happens to this as n gets very large.

When you raise most numbers to a large power the result gets bigger and bigger, except when r is a number between −1 and 1. In this case, r^n gets smaller as n increases – in fact it tends to 0.

Many other sequences and series show interesting long-term behaviour. For example if the series $\frac{4}{1} - \frac{4}{3} + \frac{4}{5} - \frac{4}{7}$ continues for ever, the result is π. Series like these are investigated in Supplementary sheet 5 'Long-term behaviour of sequences and series' on the CD-ROM.

EXAM HINT

The condition that $|r|<1$ is just as important as the formula.

We can summarise:

KEY POINT 7.7

As n increases the sum of a geometric series converges to:

$$S_\infty = \frac{u_1}{1-r} \text{ if } |r|<1$$

This is called the **sum to infinity** of the series.

When $r = 1$ the geometric sequence definitely diverges. When $r = -1$ it is uncertain whether the sequence converges or not. It might converge to 0, to u_1 or to $\frac{u_1}{2}$ depending upon how the terms are grouped. This is an example of a situation where mathematics is open to interpretation.

Worked example 7.17

The sum to infinity of a geometric series is 5. The second term is $-\frac{6}{5}$. Find the common ratio.

Express the information given as equations in terms of u_1 and r

$$S_\infty = \frac{u_1}{1-r} = 5 \quad (1)$$

$$u_2 = u_1 r = -\frac{6}{5} \quad (2)$$

Solve the equations simultaneously

From (2) $u_1 = -\frac{6}{5r}$

Substituting into (1)

$$-\frac{6}{5r(1-r)} = 5$$

$$\Leftrightarrow \quad -6 = 25(r - r^2)$$

$$\Leftrightarrow \quad 0 = 25r^2 - 25r - 6 = (5r-6)(5r+1)$$

Therefore $r = \frac{6}{5}$ or $r = -\frac{1}{5}$

Watch out for a trick! Check that the series converges

But since the sum to infinity exists, $|r|<1$ so

$$r = -\frac{1}{5}$$

Remember that some questions may focus on the condition for the sequence to converge as well as the value that it converges to.

Worked example 7.18

The geometric series $(2-x)+(2-x)^2+(2-x)^3+\ldots$ converges. What values can x take?

Identify r

$r=(2-x)$

Use the fact that the series converges

Since the series converges $|2-x|<1$

Solve the inequality

$-1<2-x<1$

$\Leftrightarrow -3<-x<-1$

Therefore $1<x<3$

Exercise 7G

1. Find the value of these infinite geometric series, or state that they are divergent.
   (a) (i) $9+3+1+\frac{1}{3}+\ldots$
   (ii) $56+8+1\frac{1}{7}+\ldots$
   (b) (i) $0.3+0.03+0.003+\ldots$
   (ii) $0.78+0.0078+0.000078+\ldots$
   (c) (i) $0.01+0.02+0.04+\ldots$
   (ii) $\frac{19}{10000}+\frac{19}{1000}+\frac{19}{100}+\ldots$
   (d) (i) $10-2+0.4-\ldots$
   (ii) $6-4+\frac{8}{3}-\ldots$
   (e) (i) $10-40+160-\ldots$
   (ii) $4.2-3.36+2.688-\ldots$
2. Find the values of x which allow these geometric series to converge.
   (a) (i) $9+9x+9x^2+\ldots$
   (ii) $-2-2x-2x^2-\ldots$
   (b) (i) $1+3x+9x^2+\ldots$
   (ii) $1+10x+100x^2+\ldots$
   (c) (i) $-2-10x-50x^2-\ldots$
   (ii) $8+24x+72x^2+\ldots$

(d) (i) $40+10x+2.5x^2+\ldots$

(ii) $144+12x+x^2+\ldots$

(e) (i) $243-81x+27x^2-\ldots$

(ii) $1-\frac{5}{4}x+\frac{25}{16}x^2-\ldots$

(f) (i) $3-\frac{6}{x}+\frac{12}{x^2}-.$

(ii) $18-\frac{9}{x}+\frac{1}{x^2}-\ldots$

(g) (i) $5+5(3-2x)+5(3-2x)^2+\ldots$

(ii) $7+\frac{7(2-x)}{2}+\frac{7(2-x)^2}{4}+\ldots$

(h) (i) $1+\left(3-\frac{2}{x}\right)+\left(3-\frac{2}{x}\right)^2+\ldots$

(ii) $1+\frac{1+x}{x}+\frac{(1+x)^2}{x^2}+\ldots$

(i) (i) $7+7x^2+7x^4+\ldots$

(ii) $12-48x^3+192x^6-\ldots$

3. Find the sum to infinity of the geometric sequence
−18, 12, −8, … *[4 marks]*

4. The first and fourth terms of a geometric series are 18 and $-\frac{2}{3}$ respectively. Find:

(a) the sum of the first n terms of the series

(b) the sum to infinity of the series. *[5 marks]*

5. A geometric sequence has all positive terms. The sum of the first two terms is 15 and the sum to infinity is 27. Find the value of

(a) the common ratio

(b) the first term. *[5 marks]*

6. The sum to infinity of a geometric sequence is 32. The sum of the first four terms is 30 and all the terms are positive.

Find the difference between the sum to infinity and the sum of the first eight terms. *[5 marks]*

7. Consider the infinite geometric series:

$$1+\left(\frac{2x}{3}\right)+\left(\frac{2x}{3}\right)^2+\ldots$$

(a) For what values of x does the series converge?

(b) Find the sum of the series if $x=1.2$. *[6 marks]*

8. The sum of an infinite geometric sequence is 13.5, and the sum of the first three terms is 13. Find the first term. *[6 marks]*

9. An infinite geometric series is given by $\sum_{k=1}^{\infty} 2(4-3x)^k$.

(a) Find the values of x for which the series has a finite sum.

(b) When $x = 1.2$, find the minimum number of terms needed to give a sum which is greater than 1.328. *[7 marks]*

10. The common ratio of the terms in a geometric series is $2x$.

(a) State the set of values of x for which the sum to infinity of the series exists.

(b) If the first term of the series is 35, find the value of x for which the sum to infinity is 40. *[6 marks]*

11. $f(x) = 1 + 2x + 4x^2 + 8x^3 \dots$ is an infinitely long expression. Evaluate:

(a) $f\left(\frac{1}{3}\right)$ (b) $f\left(\frac{2}{3}\right)$ *[6 marks]*

7H Mixed questions

Be very careful when dealing with sequences and series questions.

It is vital that you

- identify whether it is a geometric or an arithmetic sequence
- identify whether it is asking for a term in the sequence or the sum of terms in the sequence
- interpret the information given in the question into equations.

One frequently examined topic is **compound interest**. This is about savings or loans, where the interest added is a percentage of the current amount. As long as no other money is added or removed, the value of the investment will follow a geometric sequence.

If the compound interest rate is $p\%$ then this is equivalent to a ratio of $r = 1 + \frac{p}{100}$.

Exercise 7H

1. Philippa invests £1000 at 3% compound interest for 6 years.

(a) How much interest does she get paid in the 6th year?

(b) How much does she get back after 6 years? *[6 marks]*

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(c) (i) $u_{n+1} = u_n + n + 1,\ u_1 = 1$

(ii) $u_{n+2} = 2(u_{n+1} + u_n)$,
$u_1 = 1, u_2 = 2$

4. (a) (i) $u_n = 2n$

(ii) $u_n = 2n - 1$

(b) (i) $u_n = 2^n$ (ii) $u_n = 5^n$

(c) (i) $u_n = n^2$ (ii) $u_n = n^3$

(d) (i) $u_n = \dfrac{n}{n+1}$

(ii) $u_n = \dfrac{2n-1}{2^n}$

5. (a) $u_2 = 4,\ u_3 = 8,\ u_4 = 16$

(b) (i) $u_n = 2^{n-1}$

Exercise 7B

1. (a) (i) 27 (ii) 39

(b) (i) 120 (ii) $\dfrac{665}{48}$

(c) (i) $14b$ (ii) $19p$

2. (a) (i) $\sum_{2}^{43} r$ (ii) $\sum_{3}^{30} 2r$

(b) (i) $\sum_{2}^{7} \dfrac{1}{2^r}$ (ii) $\sum_{0}^{5} \dfrac{2}{3^r}$

(c) (i) $\sum_{r=2}^{10} 7ra$ (ii) $\sum_{r=0}^{19} r^b$

Exercise 7C

1. (a) (i) $u_n = 9 + 3(n-1)$

(ii) $u_n = 57 + 0.2(n-1)$

(b) (i) $u_n = 12 - (n-1)$

(ii) $u_n = 18 - \dfrac{1}{2}(n-1)$

(c) (i) $u_n = 1 + 3(n-1)$

(ii) $u_n = 9 + 10(n-1)$

(d) (i) $u_n = 4 - 4(n-1)$

(ii) $u_n = 27 - 7(n-1)$

(e) (i) $u_n = -17 + 11(n-1)$

(ii) $u_n = -32 + 10(n-1)$

2. (a) (i) 33 (ii) 29
(b) (i) 100 (ii) 226

3. (a) $a_n = 5 + 8(n-1)$ (b) 50

4. 121

5. 25

6. 17

7. $a = 2,\ b = -3$

8. (b) 456 pages

Exercise 7D

1. (a) (i) 3060 (ii) 1495
(b) (i) 9009 (ii) 23 798
(c) (i) −204 (ii) 1470
(d) (i) 667.5 (ii) 14.25

2. (a) (i) 13 (ii) 32 (iii) 53

(b) $\dfrac{x}{2}$

3. $a = 15,\ d = -8$

4. (a) $S_n = \dfrac{n}{2}(3n+1)$ (b) 30

5. 559

6. $a = 14,\ d = -8$

7. 55

8. $u_n = 6n - 5$

9. $\theta = 20°$

10. 10 300

11. 23 926

Exercise 7E

1. (a) (i) $u_n = 6 \times 2^{n-1}$

(ii) $u_n = 12 \times \left(\dfrac{3}{2}\right)^{n-1}$

(b) (i) $u_n = 20 \times \left(\dfrac{1}{4}\right)^{n-1}$

(ii) $u_n = \left(\dfrac{1}{2}\right)^{n-1}$

(c) (i) $u_n = (-2)^{n-1}$

(ii) $u_n = 5 \times (-1)^{n-1}$

(d) (i) $u_n = ax^{n-1}$

(ii) $u_n = 3\times(2x)^{n-1}$

2. (a) (i) 13 (ii) 7
(b) (i) 10 (ii) 10
(c) (i) 10 (ii) 8

3. (a) (i) 15 (ii) 31
(b) (i) 33 (ii) 17

4. 39 366

5. 10

6. 16

7. 2.5 or −1.82

8. ±384

9. $a = 7$ or -3.52

10. $a = -2, b = 4$

11. $m = 7$

Exercise 7F

1. (a) (i) 17 089 842 (ii) 2303.4375
(b) (i) 514.75 (ii) 9.487 171
(c) (i) 39 368 (ii) 9840
(d) (i) 191.953 125 or 63.984 375
(ii) 24 414 062.5 or 16 276 041.67

2. (a) (i) $r = 3$ (ii) $r = 0.2$
(b) (i) $r = -6$ (ii) $r = -0.947$

3. (a) 5 (b) $S_n = \frac{375(5^n - 1)}{4}$

4. $a = 5, r = \frac{3}{2}$

5. (a) $1 + x + x^2 + x^3$

(b) $(x-1)(x^5 + x^4 + x^3 + x^2 + x + 1)$

Exercise 7G

1. (a) (i) $\frac{27}{2}$ (ii) $\frac{196}{3}$

(b) (i) $\frac{1}{3}$ (ii) $\frac{26}{33}$

(c) (i) Divergent
(ii) Divergent

(d) (i) $\frac{25}{3}$ (ii) $\frac{18}{5}$

(e) (i) Divergent (ii) $\frac{7}{3}$

2. (a) (i) $|x| < 1$ (ii) $|x| < 1$

(b) (i) $|x| < \frac{1}{3}$ (ii) $|x| < \frac{1}{10}$

(c) (i) $|x| < \frac{1}{5}$ (ii) $|x| < \frac{1}{3}$

(d) (i) $|x| < 4$ (ii) $|x| < 12$

(e) (i) $|x| < 3$ (ii) $|x| < \frac{4}{5}$

(f) (i) $|x| > 2$ (ii) $|x| > \frac{1}{2}$

(g) (i) $1 < x < 2$ (ii) $0 < x < 4$

(h) (i) $\frac{1}{2} < x < 1$ (ii) $x < -\frac{1}{2}$

(i) (i) $|x| < 1$ (ii) $|x| < \frac{1}{\sqrt[3]{4}}$

3. $-\frac{54}{3}$

4. (a) $27 = \left(\frac{1 - \left(\frac{-1}{3}\right)^n}{2}\right)$

(b) $S_\infty = \frac{27}{2}$

5. (a) $\frac{2}{3}$ (b) 9

6. $\frac{1}{8}$

7. (a) $|x| < \frac{3}{2}$ (b) 5

8. 9

9. (a) $1 < x < \frac{5}{3}$ (b) 7

10. (a) $x < 0$ (b) $x = -3$

11. (a) 3 (b) ∞

Exercise 7H

1. (a) £34.78

(b) £1194.05

2. (a) $60 500

(b) 22 years

3. (a) 5000×1.063^n

(b) $6786.35

(c) (i) $5000 \times 1.063^n > 10000$

(ii) 12 years

4. (a) 10 (b) 23.7%

5. (a) $265.33