# Maclaurin [53 marks]

**1.** [Maximum mark: 11]

The function $f$ is defined by $f(x) = \arcsin(2x)$, where $-\frac{1}{2} \leqslant x \leqslant \frac{1}{2}$.

(a) By finding a suitable number of derivatives of $f$, find the first two non-zero terms in the Maclaurin series for $f$. [8]

---

**Markscheme**

$$f(x) = \arcsin(2x)$$

$$f'(x) = \frac{2}{\sqrt{1-4x^2}} \quad \textbf{\textit{M1A1}}$$

**Note:** Award **M1A0** for $f'(x) = \frac{1}{\sqrt{1-4x^2}}$

$$f''(x) = \frac{8x}{(1-4x^2)^{\frac{3}{2}}} \quad \textbf{\textit{A1}}$$

**EITHER**

$$f'''(x) = \frac{8(1-4x^2)^{\frac{3}{2}} - 8x\left(\frac{3}{2}(-8x)(1-4x^2)^{\frac{1}{2}}\right)}{(1-4x^2)^3} \quad \left(= \frac{8(1-4x^2)^{\frac{3}{2}} + 96x^2(1-4x^2)^{\frac{1}{2}}}{(1-4x^2)^3}\right) \quad \textbf{\textit{A1}}$$

**OR**

$$f'''(x) = 8(1-4x^2)^{-\frac{3}{2}} + 8x\left(-\frac{3}{2}(1-4x^2)^{-\frac{5}{2}}\right)(-8x) \quad \left(= 8(1-4x^2)^{-\frac{3}{2}} + 96x^2(1-4x^2)^{-\frac{5}{2}}\right)$$
     **A1**

**THEN**

substitute $x = 0$ into $f$ or any of its derivatives     **(M1)**

$$f(0) = 0, f'(0) = 2 \text{ and } f''(0) = 0 \quad \textbf{\textit{A1}}$$

$$f'''(0) = 8$$

the Maclaurin series is

$$f(x) = 2x + \frac{8x^3}{6} + \ldots \quad \left(= 2x + \frac{4x^3}{3} + \ldots\right) \quad \textbf{\textit{(M1)A1}}$$

*[8 marks]*

---

(b) Hence or otherwise, find $\lim\limits_{x \to 0} \dfrac{\arcsin(2x) - 2x}{(2x)^3}$. [3]

---

**Markscheme**

**METHOD 1**

$$\lim_{x \to 0} \frac{\arcsin(2x) - 2x}{(2x)^3} = \lim_{x \to 0} \frac{2x + \frac{4x^3}{3} + \ldots - 2x}{8x^3} \quad \textbf{\textit{M1}}$$

$$= \lim_{x \to 0} \frac{\frac{4}{3} + \ldots \text{ terms with } x}{8} \quad \textbf{\textit{(M1)}}$$

$$= \frac{1}{6} \quad \textbf{\textit{A1}}$$

**Note:** Condone the omission of $+\ldots$ in their working.

**METHOD 2**

$$\lim_{x \to 0} \frac{\arcsin(2x) - 2x}{(2x)^3} = \frac{0}{0} \text{ indeterminate form, using L'Hôpital's rule}$$

$$= \lim_{x \to 0} \frac{\frac{2}{\sqrt{1-4x^2}} - 2}{24x^2} \quad \textbf{\textit{M1}}$$

$$= \frac{0}{0} \text{ indeterminate form, using L'Hôpital's rule again}$$

$$= \lim_{x \to 0} \frac{\frac{8x}{\left(1-4x^2\right)^{\frac{3}{2}}}}{48x} \left( = \lim_{x \to 0} \frac{1}{6(1-4x^2)^{\frac{3}{2}}} \right) \quad \textbf{\textit{M1}}$$

**Note:** Award **M1** only if their previous expression is in indeterminate form.

$$= \frac{1}{6} \quad \textbf{\textit{A1}}$$

**Note:** Award **FT** for use of their derivatives from part (a).

*[3 marks]*

2.   [Maximum mark: 21]
The function $f$ is defined by $f(x) = e^{\sin x}$.

(a)   Find the first two derivatives of $f(x)$ and hence find the Maclaurin series for $f(x)$ up to and including the $x^2$ term.   [8]

Markscheme

attempting to use the chain rule to find the first derivative   *M1*

$$f'(x) = (\cos x)e^{\sin x} \quad \textbf{\textit{A1}}$$

attempting to use the product rule to find the second derivative   *M1*

$$f''(x) = e^{\sin x}\left(\cos^2 x - \sin x\right) \text{ (or equivalent)} \quad \textbf{\textit{A1}}$$

attempting to find $f(0)$, $f'(0)$ and $f''(0)$   **M1**

$f(0) = 1; f'(0) = (\cos 0)\mathrm{e}^{\sin 0} = 1; f''(0) = \mathrm{e}^{\sin 0}\left(\cos^2 0 - \sin 0\right) = 1$   **A1**

substitution into the Maclaurin formula $f(x) = f(0) + xf'(0) + \frac{x^2}{2!}f''(0) + \ldots$   **M1**

so the Maclaurin series for $f(x)$ up to and including the $x^2$ term is $1 + x + \frac{x^2}{2}$   **A1**

*[8 marks]*

(b)    Show that the coefficient of $x^3$ in the Maclaurin series for $f(x)$ is zero.   [4]

---

Markscheme

**METHOD 1**

attempting to differentiate $f''(x)$   **M1**

$f'''(x) = (\cos x)\mathrm{e}^{\sin x}\left(\cos^2 x - \sin x\right) - (\cos x)\mathrm{e}^{\sin x}\left(2\sin x + 1\right)$ (or equivalent)   **A2**

substituting $x = 0$ into **their** $f'''(x)$   **M1**

$f'''(0) = 1(1 - 0) - 1(0 + 1) = 0$

so the coefficient of $x^3$ in the Maclaurin series for $f(x)$ is zero   **AG**


**METHOD 2**

substituting $\sin x$ into the Maclaurin series for $\mathrm{e}^x$   **(M1)**

$\mathrm{e}^{\sin x} = 1 + \sin x + \frac{\sin^2 x}{2!} + \frac{\sin^3 x}{3!} + \ldots$

substituting Maclaurin series for $\sin x$   **M1**

$\mathrm{e}^{\sin x} = 1 + \left(x - \frac{x^3}{3!} + \ldots\right) + \frac{\left(x - \frac{x^3}{3!} + \ldots\right)^2}{2!} + \frac{\left(x - \frac{x^3}{3!} + \ldots\right)^3}{3!} + \ldots$   **A1**

coefficient of $x^3$ is $-\frac{1}{3!} + \frac{1}{3!} = 0$   **A1**

so the coefficient of $x^3$ in the Maclaurin series for $f(x)$ is zero   **AG**


*[4 marks]*

(c)    Using the Maclaurin series for $\arctan x$ and $\mathrm{e}^{3x} - 1$, find the Maclaurin series for $\arctan\left(\mathrm{e}^{3x} - 1\right)$ up to and including the $x^3$ term.   [6]

---

Markscheme

substituting $3x$ into the Maclaurin series for $\mathrm{e}^x$    *M1*

$$\mathrm{e}^{3x} = 1 + 3x + \frac{(3x)^2}{2!} + \frac{(3x)^3}{3!} + \dots \quad \textit{A1}$$

substituting $\left(\mathrm{e}^{3x} - 1\right)$ into the Maclaurin series for $\arctan x$    *M1*

$$\arctan\left(\mathrm{e}^{3x} - 1\right) = \left(\mathrm{e}^{3x} - 1\right) - \frac{\left(\mathrm{e}^{3x}-1\right)^3}{3} + \frac{\left(\mathrm{e}^{3x}-1\right)^5}{5} - \dots$$

$$= \left(3x + \frac{(3x)^2}{2!} + \frac{(3x)^3}{3!} + \dots\right) - \frac{\left(3x + \frac{(3x)^2}{2!} + \frac{(3x)^3}{3!} + \dots\right)^3}{3} + \dots \quad \textit{A1}$$

selecting correct terms from above    *M1*

$$= \left(3x + \frac{(3x)^2}{2!} + \frac{(3x)^3}{3!}\right) - \frac{(3x)^3}{3}$$

$$= 3x + \frac{9x^2}{2} - \frac{9x^3}{2} \quad \textit{A1}$$

*[6 marks]*

(d)    Hence, or otherwise, find $\displaystyle \lim_{x \to 0} \frac{f(x)-1}{\arctan(\mathrm{e}^{3x}-1)}$.    [3]

Markscheme

***METHOD 1***

substitution of **their** series    *M1*

$$\lim_{x \to 0} \frac{x + \frac{x^2}{2} + \dots}{3x + \frac{9x^2}{2} + \dots} \quad \textit{A1}$$

$$= \lim_{x \to 0} \frac{1 + \frac{x}{2} + \dots}{3 + \frac{9x}{2} + \dots}$$

$$= \frac{1}{3} \quad \textit{A1}$$

***METHOD 2***

use of l'Hôpital's rule    *M1*

$$\lim_{x \to 0} \frac{(\cos x)\mathrm{e}^{\sin x}}{\frac{3\mathrm{e}^{3x}}{1 + \left(\mathrm{e}^{3x} - 1\right)^2}} \quad \text{(or equivalent)} \quad \textit{A1}$$

$$= \frac{1}{3} \quad \textit{A1}$$

*[3 marks]*

**3.** [Maximum mark: 21]

The function $f$ is defined by $f(x) = \mathrm{e}^x \sin x$, where $x \in \mathbb{R}$.

The function $g$ is defined by $g(x) = \mathrm{e}^x \cos x$, where $x \in \mathbb{R}$.

(a)    Find the Maclaurin series for $f(x)$ up to and including the $x^3$ term.    [4]

---

Markscheme

**METHOD 1**

recognition of both known series    *(M1)*

$\mathrm{e}^x = 1 + \frac{x}{1!} + \frac{x^2}{2!} + \dots$ and $\sin x = x - \frac{x^3}{3!} + \frac{x^5}{5!} + \dots$

attempt to multiply the two series up to and including $x^3$ term    *(M1)*

$\mathrm{e}^x \sin x = \left(1 + \frac{x}{1!} + \frac{x^2}{2!} + \dots\right)\left(x - \frac{x^3}{3!} + \frac{x^5}{5!} + \dots\right)$

$= x - \frac{x^3}{3!} + x^2 + \frac{x^3}{2!} + \dots$    *(A1)*

$\mathrm{e}^x \sin x = x + x^2 + \frac{1}{3}x^3 + \dots$    *A1*

**METHOD 2**

$f(x) = \mathrm{e}^x \sin x$

$f\prime(x) = \mathrm{e}^x \cos x + \mathrm{e}^x \sin x$    *A1*

$f\prime\prime(x) = \mathrm{e}^x \cos x - \mathrm{e}^x \sin x + \mathrm{e}^x \sin x + \mathrm{e}^x \cos x \ (= 2\mathrm{e}^x \cos x)$

$f\prime\prime\prime(x) = 2\mathrm{e}^x \cos x - 2\mathrm{e}^x \sin x$

$f\prime\prime(x) = 2\mathrm{e}^x \cos x$ and $f\prime\prime\prime(x) = 2\mathrm{e}^x(\cos x - \sin x)$    *A1*

substitute $x = 0$ into $f$ or its derivatives to obtain Maclaurin series    *(M1)*

$\mathrm{e}^x \sin x = 0 + \frac{x}{1!} \times 1 + \frac{x^2}{2!} \times 2 + \frac{x^3}{3!} \times 2 + \dots$

$\mathrm{e}^x \sin x = x + x^2 + \frac{1}{3}x^3 + \dots$    *A1*

*[4 marks]*

---

(b)    Hence, find an approximate value for $\int_0^1 \mathrm{e}^{x^2} \sin\left(x^2\right) \mathrm{d}\,x$.    [4]

---

Markscheme

$e^{x^2} \sin\left(x^2\right) = x^2 + x^4 + \frac{1}{3}x^6 + \ldots$    *(A1)*

substituting their expression and attempt to integrate    *M1*

$\int_0^1 e^{x^2} \sin\left(x^2\right) \mathrm{d}\,x \approx \int_0^1 \left(x^2 + x^4 + \frac{1}{3}x^6\right) \mathrm{d}\,x$

**Note:** Condone absence of limits up to this stage.

$= \left[\frac{x^3}{3} + \frac{x^5}{5} + \frac{x^7}{21}\right]_0^1$    *A1*

$= \frac{61}{105}$    *A1*

*[4 marks]*

(c)    Show that $g(x)$ satisfies the equation $g''(x) = 2(g'(x) - g(x))$.    [4]

Markscheme

attempt to use product rule at least once    *M1*

$g'(x) = e^x \cos x - e^x \sin x$    *A1*

$g''(x) = e^x \cos x - e^x \sin x - e^x \sin x - e^x \cos x (= -2e^x \sin x)$    *A1*

**EITHER**

$2(g'(x) - g(x)) = 2(e^x \cos x - e^x \sin x - e^x \cos x) = -2e^x \sin x$    *A1*

**OR**

$g''(x) = 2(e^x \cos x - e^x \sin x - e^x \cos x)$    *A1*

**THEN**

$g''(x) = 2(g'(x) - g(x))$    *AG*

**Note:** Accept working with each side separately to obtain $-2e^x \sin x$.

*[4 marks]*

(d)    Hence, deduce that $g^{(4)}(x) = 2(g'''(x) - g''(x))$.                                    [1]

**Markscheme**

$g'''(x) = 2(g''(x) - g'(x))$    *A1*

$g^{(4)}(x) = 2(g'''(x) - g''(x))$    *AG*

**Note:** Accept working with each side separately to obtain $-4e^x \cos x$.

*[1 mark]*

(e)    Using the result from part (c), find the Maclaurin series for $g(x)$ up to and including the $x^4$ term.    [5]

**Markscheme**

attempt to substitute $x = 0$ into a derivative    *(M1)*

$g(0) = 1,\ g'(0) = 1,\ g''(0) = 0$    *A1*

$g'''(0) = -2,\ g^{(4)}(0) = -4$    *(A1)*

attempt to substitute into Maclaurin formula    *(M1)*

$g(x) = 1 + x - \frac{2}{3!}x^3 - \frac{4}{4!}x^4 + \ldots \left(= 1 + x - \frac{1}{3}x^3 - \frac{1}{6}x^4 + \ldots\right)$    *A1*

**Note:** Do not award any marks for approaches that do not use the part (c) result.

*[5 marks]*

(f)    Hence, or otherwise, determine the value of $\lim\limits_{x \to 0} \frac{e^x \cos x - 1 - x}{x^3}$.    [3]

**Markscheme**

**METHOD 1**

$\lim\limits_{x \to 0} \frac{e^x \cos x - 1 - x}{x^3} = \lim\limits_{x \to 0} \frac{\left(1 + x - \frac{1}{3}x^3 - \frac{1}{6}x^4 + \ldots\right) - 1 - x}{x^3}$    *M1*

$= \lim\limits_{x \to 0}\left(-\frac{1}{3} - \frac{1}{6}x + \ldots\right)$    *(A1)*

$= -\frac{1}{3}$    *A1*

**Note:** Condone the omission of $+\ldots$ in their working.


**METHOD 2**

$\lim\limits_{x \to 0} \frac{\mathrm{e}^x \cos x - 1 - x}{x^3} = \frac{0}{0}$ indeterminate form, attempt to apply l'Hôpital's rule   ***M1***

$= \lim\limits_{x \to 0} \frac{\mathrm{e}^x \cos x - \mathrm{e}^x \sin x - 1}{3x^2} \left( = \lim\limits_{x \to 0} \frac{g\prime(x) - 1}{3x^2} \right)$

$= \frac{0}{0}$, using l'Hôpital's rule again

$= \lim\limits_{x \to 0} \frac{-2\mathrm{e}^x \sin x}{6x} \left( = \lim\limits_{x \to 0} \frac{g\prime\prime(x)}{6x} \right)$

$= \frac{0}{0}$, using l'Hôpital's rule again

$= \lim\limits_{x \to 0} \frac{-2\mathrm{e}^x \sin x - 2\mathrm{e}^x \cos x}{6} \left( = \lim\limits_{x \to 0} \frac{g\prime\prime\prime(x)}{6} \right)$   ***A1***

$= -\frac{1}{3}$   ***A1***


*[3 marks]*