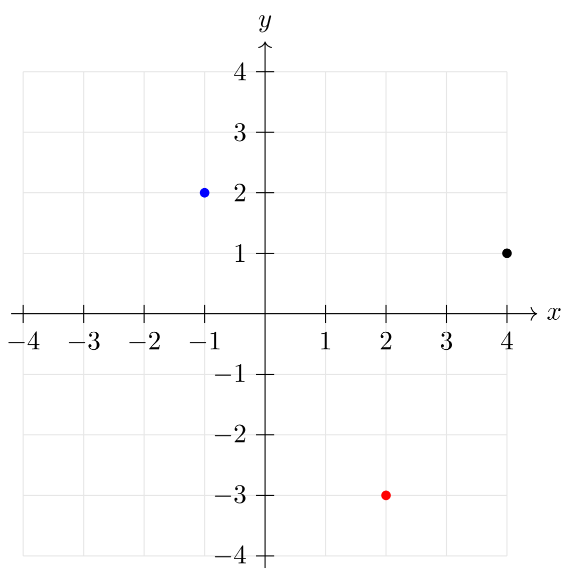
Chapter 1

Coordinate geometry

1.1 Coordinate system

A point in a 2D coordinate system (coordinate plane) can be uniquely identified by 2 coordinates. We usually refer to the first coordinate as the x-coordinate and the second coordinate as the y-coordinate. For instance the point $(4, 1)$ has x-coordinate equal to 4 and y-coordinate equal to 1. It is represented with a black point below.



The blue and red points represent $(-1, 2)$ and $(2, -3)$ respectively.

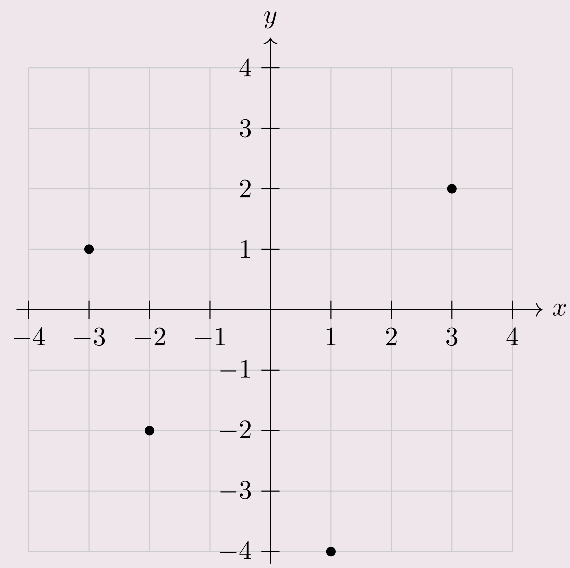
Exercise 3.1.1a Label the following points on a 2d coordinate system:

(a) $(1, 1)$ (b) $(-2, -3)$ (c) $(3, -4)$

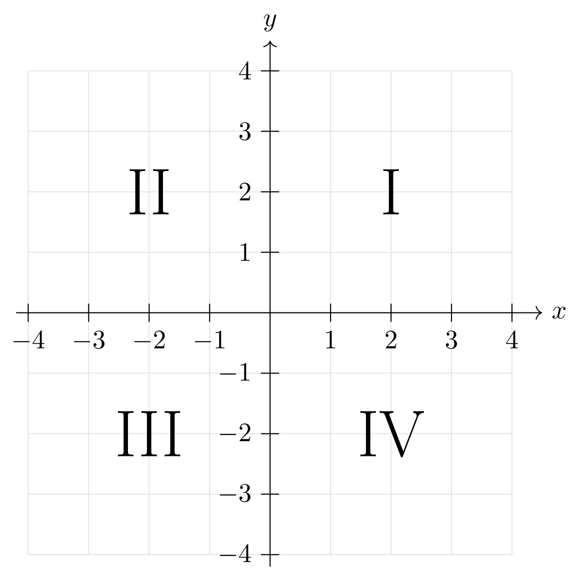
(d) $(0, 4)$ (e) $(-3, 0)$ (f) $(0, 0)$

(g) $(\pi, 2)$ (h) $(\sqrt{2}, -2\sqrt{2})$ (i) $(5 - 2\sqrt{6}, 3\sqrt{3} - 5)$

Exercise 3.1.1b Write down the coordinates of the following points (all coordinates are integers):



A 2D coordinate plane can be divided into 4 quadrants (usually denoted with Roman numerals):



In the first quadrant both coordinates are positive. In the second quadrant we have $x < 0$ and $y > 0$. In quadrant III: $x < 0$ and $y < 0$. Finally in quadrant IV: $x > 0$ and $y < 0$.

Exercise 3.1.1c State in which quadrant the following points are located:

(a) $(2, -3)$

(b) $(-1, -20)$

(c) $(2 - \sqrt{3}, 4\sqrt{5} - 9)$

(d) $((-2)^3, (-2)^4)$

(b) $(3^0, \sqrt{7} - 3)$

(c) $(|1 - \sqrt{5}|, 2\sqrt[3]{3} - 5)$

Note that the axes are in general not considered to be parts of any quadrant. So for example the point $(0, 3)$ lies on the y-axis (specifically the positive part of the y-axis) and not in any of the quadrants.

Exercise 3.1.2a In a 2D video game, a character moves on a 10×10 coordinate grid. The character begins at the spawn point $(0, 0)$.

(a) The character moves to collect a power-up at $(3, -1)$, then to a checkpoint at $(-2, -4)$. Plot and label these three points on a coordinate plane.

(b) To unlock a secret level, the character must reach a point where $x + y = 3$. Find two points with integer coordinates that satisfy this equation and plot them.

(c) The game restricts the character from entering Quadrant IV. Which of the points from parts (a) and (b) are in Quadrant IV, if any?

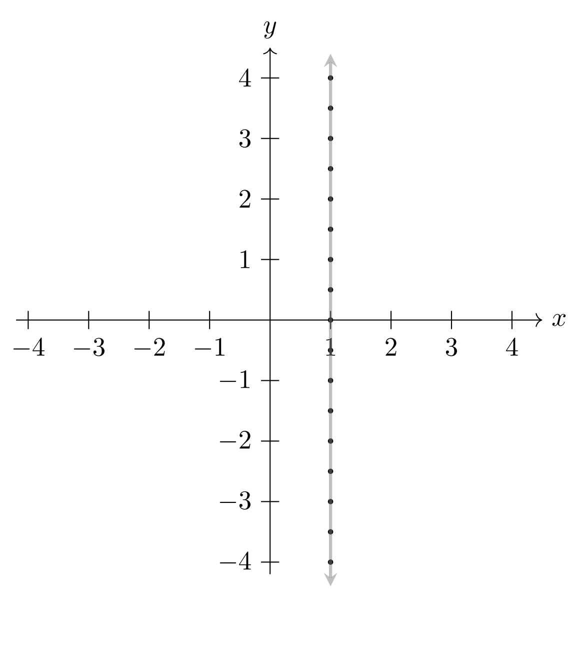
Exercise 3.1.2b A smart vacuum cleaner navigates a room on an 8×8 coordinate grid. It starts at its docking station at $(0, 0)$.

(a) The vacuum moves to clean a spill at $(1, 3)$, then to a pet bed at $(2, -2)$. Plot and label these three points on a coordinate plane.

(b) To avoid obstacles, the vacuum must stop at a point where $y = 2$. List three points with integer x-coordinates that satisfy this equation and plot them.

(c) If the vacuum is programmed to stay out of Quadrant I, which of the points from parts (a) and (b) should it avoid?

Suppose we would like to label all points with x-coordinate equal to 1. These points include $(1, -2)$, $(1, 0)$ and $(1, \sqrt{5})$. Of course there are infinitely many points with the first coordinate equal to 1. We can label some of them on the coordinate plane as follows:



Note that the diagram includes 17 points that have the first coordinate equal to 1, that is they satisfy the equation $x = 1$. The diagram also include a line that indicates **all** points satisfying the equation $x = 1$. Note that the line extends indefinitely as indicated by the arrows.

Exercise 3.1.3 On a 2d coordinate system label all points that satisfy the following equations:

(a) $x = -2$ (b) $y = 4$ (c) $2x = 6$

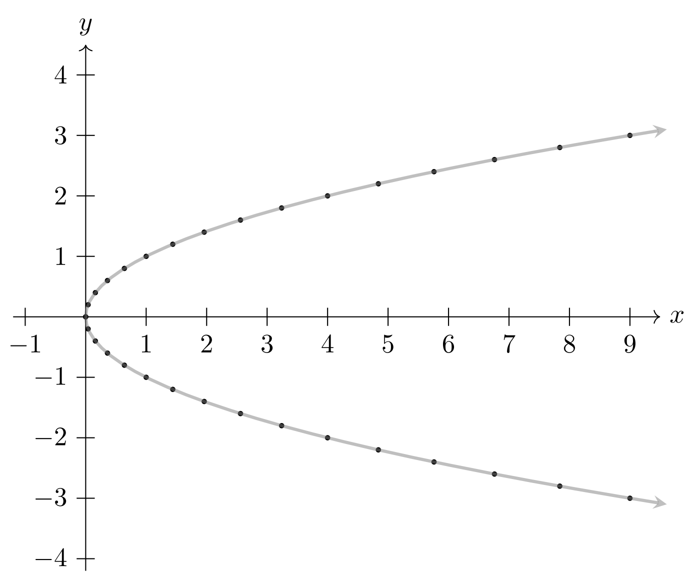
(d) $x = y$ (e) $x + y = 2$ (f) $x - y = 1$

(g) $2x + y = 3$ (h) $3x - 2y = 6$ (i) $\frac{x}{2} - y = 2$

Consider the equation $x = y^2$. Points whose first coordinate is the square of the second coordinate satisfy this equation. Examples of these points include $(1,1)$, $(4,2)$, $(9,-3)$. These points satisfy the given equation because $1 = 1^2$ $4 = 2^2$ and $9 = (-3)^2$. Of course there are many more points that satisfy the equation. We can create a table of these points by varying the possible values of the second coordinate:

x	9	7.84	6.76	...	0.04	0	0.04	...	4	4.84	5.76	...
y	-3	-2.8	-2.6	...	-0.2	0	0.2	...	2	2.2	2.4	...

We can mark such points on the coordinate plane:



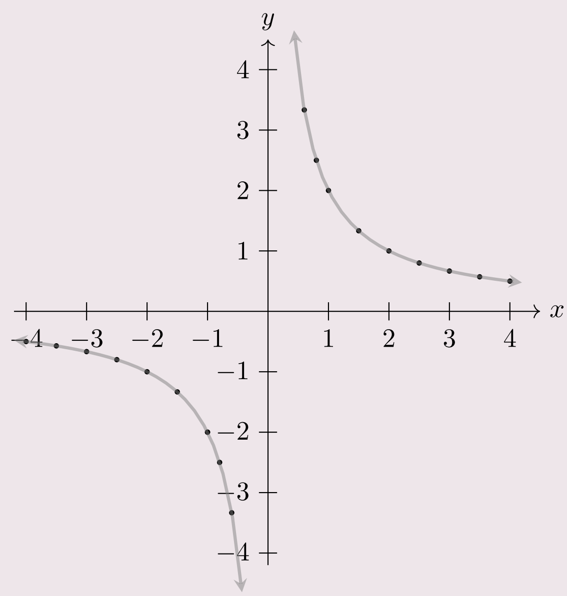
Note that the above diagram includes 31 points that satisfy the equation $x = y^2$, these are, of course, not all points that satisfy the equation. The gray curve includes all points that satisfy the equation. Note that the curve extends indefinitely as indicated by arrows.

Worked example 3.1.4.

Label several points on the coordinate plane that satisfy the equation $xy = 2$ and hence draw a curve that contains all points that satisfy this equation.

Examples of points satisfying $xy = 2$ include $(2, 1)$, $(1, 2)$, but also $(4, \frac{1}{2})$, $(-3, -\frac{2}{3})$ and $(-1, -2)$. Note that the point $(2, 1)$ satisfies the equation $xy = 2$, because $2 \times 1 = 2$. We can include more points by varying one of the coordinates:

x	4	-3	-2	-1	$-\frac{1}{2}$	0	$\frac{1}{2}$	1	2	3	4	...
y	$\frac{1}{2}$	$-\frac{2}{3}$	-1	-2	-4	–	4	2	1	$\frac{2}{3}$	$\frac{1}{2}$	...



The diagram below includes several points satisfying the equation $xy = 2$ and also a curve that indicates all points satisfying this equation. Note that no points with x-coordinate or y-coordinate 0 satisfies this equation.

Exercise 3.1.4 On a 2d coordinate system label all points that satisfy the following equations:

(a) $y = x^2$

(b) $y = \sqrt{x}$

(c) $y = -\sqrt{x}$

(d) $y = 2^x$

(e) $x = 2^y$

(f) $y = 3^x$

(g) $y = |x|$

(h) $x = |y|$

(i) $x^2 = y^2$

Worked example 3.1.5.

Label all points on the coordinate plane that satisfy the equation:

$$x^2 + 2x - 2y = xy$$

We start by moving all terms to one side:

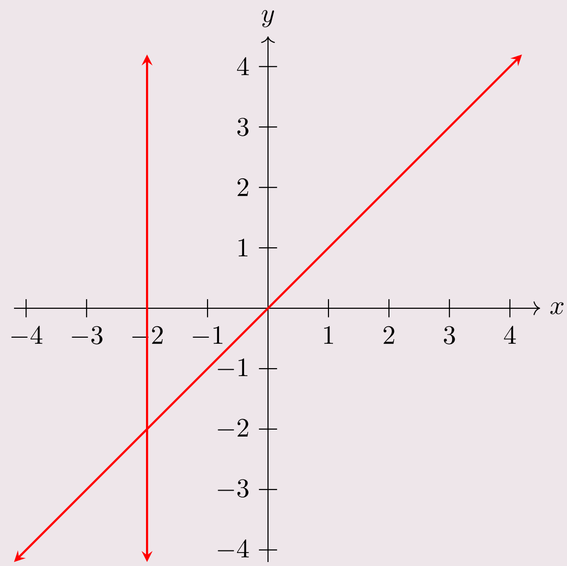
$$x^2 + 2x - 2y - xy = 0$$

We can now factorize the left hand side by grouping:

$$x(x + 2) - y(2 + x) = 0$$

$$(x - y)(x + 2) = 0$$

So either $x = y$ or $x = -2$. We can now draw all the points that satisfy the above equations:



Any point on either of the two lines satisfies the given equation.

Exercise 3.1.5. On a 2d coordinate system label all points that satisfy the following equations:

(a) $xy + 3x - y = 3$

(b) $xy + y^2 - 2x = 2y$

(c) $x^3 + y = x^2 + yx$

(d) $x^2 + xy^2 + 3y^2 + 3x = 0$

(e) $2^{x+y} - x2^x - y2^y + xy = 0$

(f) $x^2 2^x - y^2 = yx^2 - y2^x$

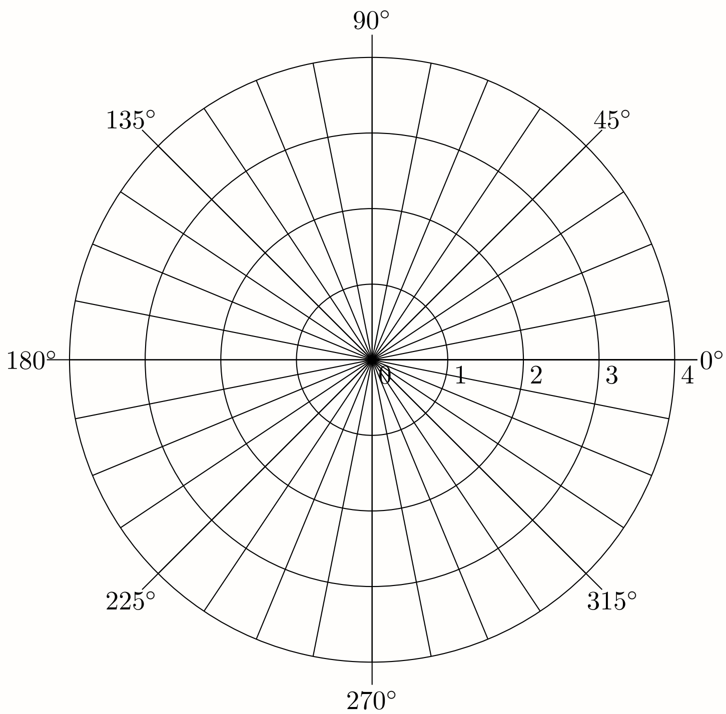
Polar coordinates

Imagine you're at the origin facing in the direction of the positive x-axis and you want to reach the point with coordinate $(1, 1)$. One way of doing this is to move 1 unit in the positive x-direction and then move 1 unit in the positive y-direction. Another way would be to turn anti-clockwise by an angle θ and move r units forward.

1. Write down the value of θ and r.

   r and θ are called the **polar** coordinates of the point $(1, 1)$. The coordinates $(1, 1)$ are called its Cartesian coordinates.

   The polar grid is often represented as follows:



2. Write down the polar coordinates of the following points expressed in Cartesian coordinates:
   (a) $(0, 3)$ (b) $(-2, 0)$ $(-\sqrt{2}, \sqrt{2})$

3. Write down the Cartesian coordinates of the following points expressed in polar coordinates (the first coordinate is r, the second is θ):
   (a) $(1, 90°)$ (b) $(3, 270°)$ $(2, 45°)$

4. Mark the points from questions 2. and 3. on the polar grid above.

SHORT TEST

1. [2 *points*]

State in which quadrant the following points are located:

(a) $(4-\sqrt{17}, (-2)^3)$

(b) $(|2-\sqrt{11}|, -2^2)$

2. [6 *points*]

A smart vacuum cleaner navigates a room on an 8×8 coordinate grid. It starts at its docking station at $(1, 1)$.

(a) The vacuum moves to clean a spill at $(3, 4)$, then to a toy at $(0, -1)$. Plot and label these three points on a coordinate plane.

(b) To avoid furniture, the vacuum must stop at a point where $x + y = 3$. List three points with integer x-coordinates that satisfy this equation and plot them.

(c) If the vacuum is programmed to stay out of Quadrant IV, which of the points from parts (a) and (b) should it avoid?

3. [4 *points*]

Sketch the set of points satisfying the following equations:

(a) $x + y = 1$

(b) $y = x^2 - 1$

SHORT TEST
SOLUTIONS

1. [2 *points*]
State in which quadrant the following points are located:

(a) $(4-\sqrt{17}, (-2)^3)$

(b) $(|2-\sqrt{11}|, -2^2)$

(a) $4-\sqrt{17} < 0$ and $(-2)^3 = -8 < 0$, so III quadrant.

(b) $|2-\sqrt{11}| > 0$ and $-2^2 = -4 < 0$, so IV quadrant.

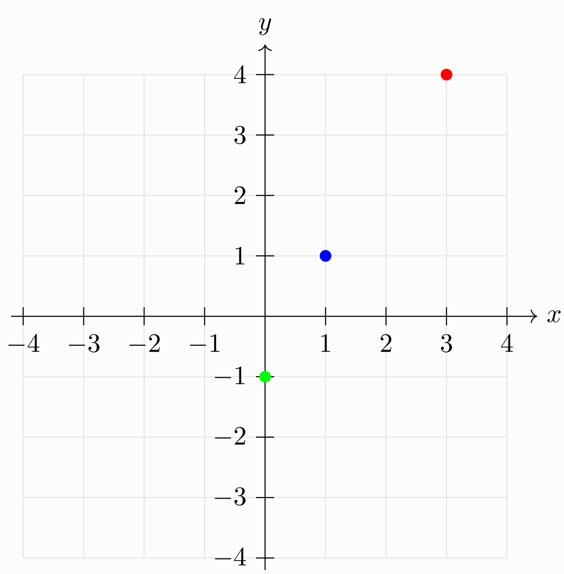
2. [6 *points*]
A smart vacuum cleaner navigates a room on an 8×8 coordinate grid. It starts at its docking station at $(1,1)$.

(a) The vacuum moves to clean a spill at $(3,4)$, then to a toy at $(0,-1)$. Plot these three points on a coordinate plane.

(b) To avoid furniture, the vacuum must stop at a point where $x+y=3$. List three points with integer x-coordinates that satisfy this equation and plot them.

(c) If the vacuum is programmed to stay out of Quadrant IV, which of the points from parts (a) and (b) should it avoid?

(a)



blue - docking station, red - spill, green - toy.

(b) For example $(1,2), (3,0)$ and $(4,-1)$.

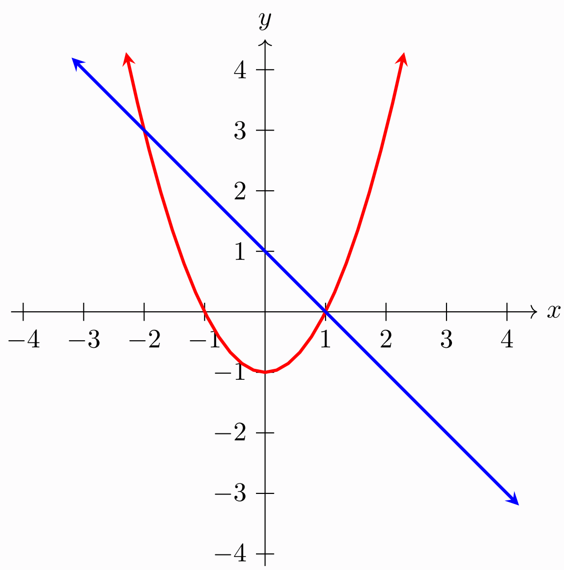
(c) In the IV quadrant we have $x > 0$ and $y < 0$, so only $(4,-1)$ is be avoided.

3. [4 *points*]

Sketch the set of points satisfying the following equations:

(a) $x + y = 1$

(b) $y = x^2 - 1$

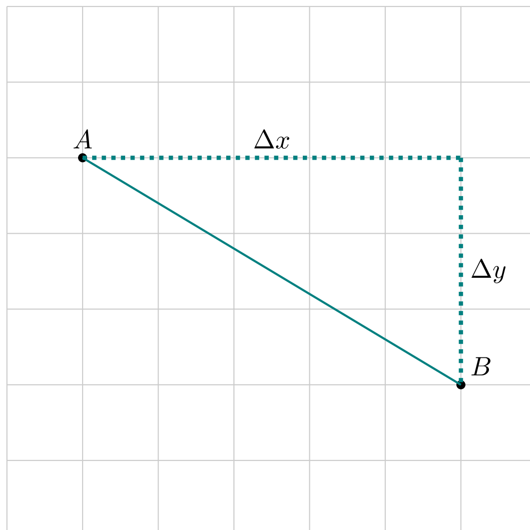


(a) in blue, (b) in red.

1.2 Line segments

A line segment between points A and B is denoted as $\overline{AB}$. Length of a line segment $\overline{AB}$, i.e. the distance between points A (x_A, y_A) and B (x_B, y_B), can be calculated using Pythagora's theorem:

$$|AB| = \sqrt{(\Delta x)^2 + (\Delta y)^2} = \sqrt{(x_B - x_A)^2 + (y_B - y_A)^2}$$



Note that Δx and Δy denote the change in x and y-coordinates, respectively. Δx is the horizontal change and Δy is the vertical change.

Worked example 3.2.1.

Find the length of the line segment $\overline{AB}$, where $A = (-1, -3)$ and $B = (4, -1)$.

The x-coordinate changes from -1 to 4, so the change in the x-coordinate is:

$$\Delta x = 4 - (-1) = 5$$

The y-coordinate changes from -3 to -1, so the change in the y-coordinate is:

$$\Delta y = -1 - (-3) = 2$$

So the distance from A to B (the length of $\overline{AB}$) is:

$$|AB| = \sqrt{5^2 + 2^2} = \sqrt{29}$$

Exercise 3.2.1. Find the length of the line segments $\overline{AB}$, where:

(a) $A = (-2, 1) \quad B = (-3, 5)$

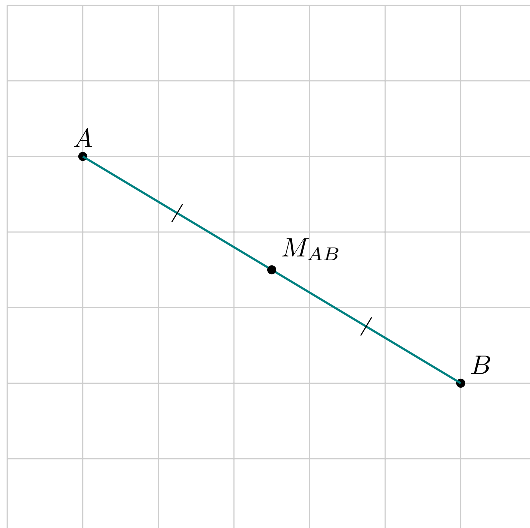
(b) $A = (3, 4) \quad B = (2, 0)$

(c) $A = (0, 5) \quad B = (\sqrt{2}, 1)$

(d) $A = (a, 2a) \quad B = (2a - 1, a + 2)$

The **midpoint** of line segment $\overline{AB}$ is a point on $\overline{AB}$, which is equidistant from A and B. For $A(x_A, y_A)$ and $B(x_B, y_B)$ the midpoint of $\overline{AB}$ is:

$$M_{AB} = \left(\frac{x_A + x_B}{2}, \frac{y_A + y_B}{2}\right)$$



Worked example 3.2.2.

Find the midpoint of the line segment $\overline{AB}$, where $A(-2, 4)$ and $B(4, 0)$

We have:

$$M_{AB} = \left(\frac{-2+4}{2}, \frac{4+0}{2}\right) = (1, 2)$$

Exercise 3.2.2a Find the midpoint of the line segments $\overline{AB}$, where:

(a) $A = (-6, 1) \quad B = (2, 3)$

(b) $A = (\sqrt{2}, \sqrt{3}) \quad B = (3\sqrt{2}, -5\sqrt{3})$

(c) $A = (t, 3t + 1) \quad B = (1 - t, t - 3)$

(d) $A = (a + b, a - b) \quad B = (3a - 5b, b - 3a)$

Exercise 3.2.2b Find the coordinate of point A given that:

(a) $B = (-3, 2) \quad M_{AB} = (-1, 1)$

(b) $B = (0, 5) \quad M_{AB} = (-1, 2)$

(c) $B = (2p, p - 1) \quad M_{AB} = (p, 3p)$

(d) $B = (t + 2, 3 - t) \quad M_{AB} = (t - 1, 2t)$

Worked example 3.2.3.

The line segment $\overline{AB}$ with $A(2, k)$ has a midpoint $M_{AB} = (5, 3)$. Given that the length of $\overline{AB}$ is $\sqrt{40}$, find the possible values of k and the coordinate of B.

Using the midpoint formula we get that:

$$\frac{2 + x_B}{2} = 5 \quad \text{and} \quad \frac{k + y_B}{2} = 3$$

Which gives the coordinates of B as $(8, 6 - k)$. Now we have:

$$\Delta x = 6 \quad \text{and} \quad \Delta y = 6 - 2k$$

So using the length of $\overline{AB}$ we get:

$$\sqrt{6^2 + (6 - 2k)^2} = \sqrt{40}$$

Which gives:

$$(6 - 2k)^2 = 4$$

So:

$$6 - 2k = \pm 2$$

And finally we get $k = 2$ or $k = 4$, so B has coordinates $(8, 4)$ or $(8, 2)$.

Exercise 3.2.3a The line segment $\overline{AB}$ has endpoints $A(-1, 3)$ and $B(4, b)$. If the length of AB is $\sqrt{41}$, find the possible coordinates of B.

Exercise 3.2.3b The line segment $\overline{AB}$ has endpoints $A(1, -2)$ and $B(5, t + 2)$. If the coordinates of the midpoint are $M(3, t - 1)$, find the value of t and length of $\overline{AB}$.

Exercise 3.2.3c The line segment $\overline{AB}$ has endpoint $A(-2, k)$ and midpoint $M(-3, -1)$. If the length of AB is $2\sqrt{5}$, find the possible values of k and the coordinates of B.

Exercise 3.2.3d The line segment $\overline{AB}$ has endpoint $A(2, 1)$ and midpoint $M(q, 2)$. If the length of AB is $2\sqrt{10}$, find the possible values of q and the coordinates of B.

Exercise 3.2.3e The line segment $\overline{AB}$ has endpoint $A(a, 1)$ and midpoint $M(2, a)$. If the length of AB is $2\sqrt{5}$, find the possible values of a and the coordinates of B.

Worked example 3.2.4.

The line segment $\overline{AB}$ has endpoint $A(1,5)$ and midpoint $M(2,k)$. The coordinates of point B satisfy the equation $x+y=0$. Find the value of k and the length of $\overline{AB}$

Using the midpoint we must have that:

$$\frac{1+x_B}{2}=2$$

which gives $x_B=3$, so B has coordinates $(3,y_B)$. Now they must satisfy the equation $x+y=0$, so:

$$3+y_B=0$$

which gives $y_B=-3$ and the coordinates of B are $(3,-3)$. We can now calculate the y-coordinate of the midpoint:

$$k=\frac{5+(-3)}{2}=1$$

We can also calculate the length of $\overline{AB}$. We have $\Delta x=2$ and $\Delta y=8$, so:

$$|AB|=\sqrt{2^2+8^2}=\sqrt{68}=2\sqrt{17}$$

Exercise 3.2.4a The line segment $\overline{AB}$ has endpoint $A(2,1)$ and midpoint $M(3,m)$. The coordinates of point B satisfy the equation $x+y=2$. Find the value of m and the length of $\overline{AB}$.

Exercise 3.2.4b The line segment $\overline{AB}$ has endpoint $A(0,2)$ and midpoint $M(q,5)$. The coordinates of point B satisfy the equation $2x+y=4$. Find the value of q and the length of $\overline{AB}$.

Exercise 3.2.4c The line segment $\overline{AB}$ has endpoint $A(-1,1)$ and midpoint $M(2,r)$. The coordinates of point B satisfy the equation $y=x^2$. Find the value of r and the length of $\overline{AB}$.

Exercise 3.2.4d The line segment $\overline{AB}$ has endpoint $A(\sqrt{2},\sqrt{3})$ and midpoint $M(2\sqrt{2},s)$. The coordinates of point B satisfy the equation $3y^2=x^2$. Find the possible values s.

Exercise 3.2.4e The line segment $\overline{AB}$ has endpoint $B(1,-2)$ and midpoint $M(t,1)$. The coordinates of point A satisfy the equation $xy=5$. Find the value of t.

The **gradient** of a line segment $\overline{AB}$ is the ratio of the change in y-coordinate (Δy) to the change in x-coordinate (Δx). The gradient is usually denoted with the letter m.

$$m = \frac{\Delta y}{\Delta x} = \frac{y_B - y_A}{x_B - x_A}$$

Note that the gradient can be thought of as *rise* over *run*, where *rise* refers to the vertical change and *run* refers to the horizontal change.

Note that for horizontal line segments the gradient is 0, since $\Delta y = 0$ and for vertical line segments the gradient is undefined, since $\Delta x = 0$.

The gradient of $\overline{AB}$ is of course equal to the gradient of $\overline{BA}$. This follows from the formula, but also from the fact the line segments have no direction and hence $\overline{AB}$ is the same line segment as $\overline{BA}$.

Exercise 3.2.5 Find the gradient the line segments $\overline{AB}$, where:

(a) $A = (1, 5) \quad B = (2, 3)$

(b) $A = (-1, -5) \quad B = (3, 1)$

(c) $A = (2, 5) \quad B = (4, 5)$

(d) $A = (1, -5) \quad B = (1, 4)$

Worked example 3.2.6

The line segment $\overline{AB}$ has endpoints $A(-1, 3)$ and $B(3, b)$. Find b, given that the gradient of this line segment is $-\frac{1}{2}$.

We have $\Delta x = 3 - (-1) = 4$ and $\Delta y = b - 3$, so, using the gradient, we get:

$$\frac{b-3}{4} = -\frac{1}{2}$$

which gives $b = 1$.

Exercise 3.2.6 Find the unknown coordinates in the following examples:

(a) $A = (-1, -2) \quad B = (3, a) \quad m = 1$

(b) $A = (4, 1) \quad B = (b, 2) \quad m = \frac{1}{4}$

(c) $A = (c, 3) \quad B = (3, 4) \quad m = 2$

(d) $A = \left(\frac{1}{2}, -\frac{1}{3}\right) \quad B = (1, d) \quad m = 3$

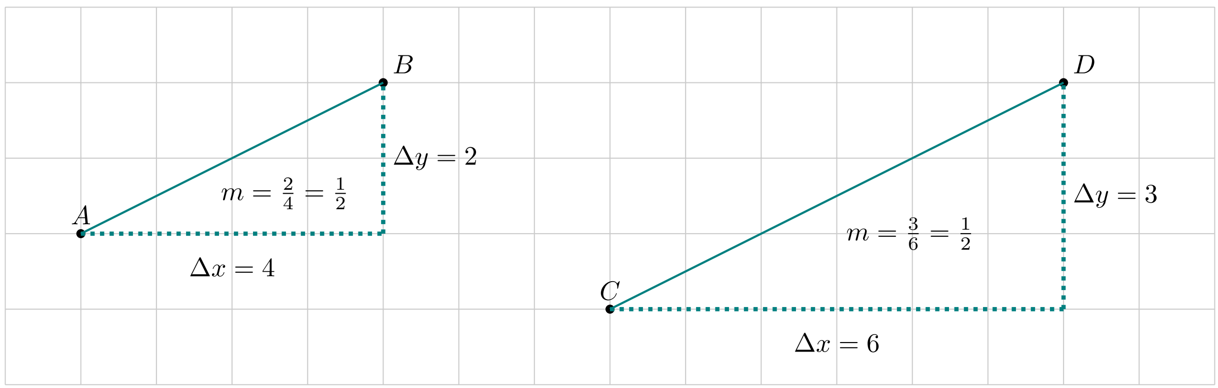
(e) $A = (e, 2e) \quad B = (3, 4) \quad m = 3$

(f) $A = (-1, f) \quad B = (2f, 3) \quad m = -1$

(g) $A = (1, 1) \quad B = (g, g^2) \quad m = 3$

(h) $A = (1, 1) \quad B = \left(h, \frac{1}{h}\right) \quad m = -\frac{1}{2}$

Two line segments are parallel, if their gradients are the same:



Exercise 3.2.7. Find unkown coordinates given that the line segments $\overline{AB}$ and $\overline{CD}$ are parallel.

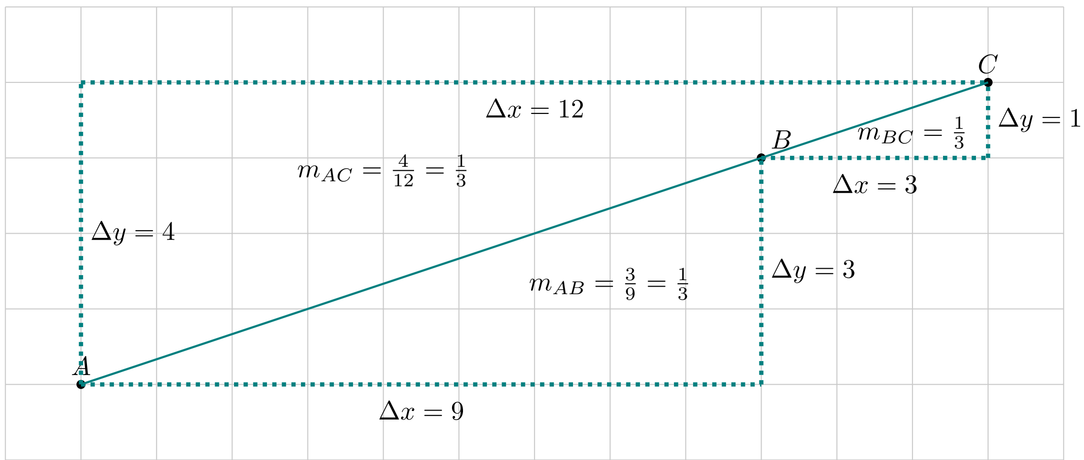
(a) $A(-2,3)$, $B(1,4)$, $C(-5,1)$, $D(1,d)$

(b) $A(0,-2)$, $B(2,-6)$, $C(c,7)$, $D(1,1)$

(c) $A(a,0)$, $B(2,3)$, $C(-2,-1)$, $D(\sqrt{2},\sqrt{2})$

(d) $A(3,1)$, $B(\sqrt{3},b)$, $C(1,2)$, $D(\sqrt{3},3)$

Three points A, B and C are collinear (lie on the same line, or there is a line segment containing all three), if the gradient of $\overline{AB}$ is equal to the gradient of $\overline{AC}$ (it is also equal to the gradient of $\overline{BC}$).



Exercise 3.2.8. Find unknown coordinates given that the points A, B and C are collinear.

(a) $A(0,1)$, $B(3,7)$, $C(4,p)$

(b) $A(-2,-4)$, $B(1,q)$, $C(2,-2)$

(c) $A(r,0)$, $B(3,5)$, $C(-3,-10)$

(d) $A(\sqrt{2},2\sqrt{2})$, $B(5\sqrt{2},\sqrt{2})$, $C(4\sqrt{2},s)$

Worked example 3.2.9.

The line segment $\overline{AB}$ has endpoint $A(-1, 4)$. Find the coordinates of B, given that the gradient of this line segment is -1 and the length $\overline{AB}$ is $3\sqrt{2}$.

We have $\Delta x = x_B + 1$ and $\Delta y = y_B - 4$. Using the gradient formula we get that:

$$\frac{y_B - 4}{x_B + 1} = -1$$

which gives $y_B = 3 - x_b$. Now using the distance formula:

$$\sqrt{(x_B + 1)^2 + (y_B - 4)^2} = 3\sqrt{2}$$

$$\sqrt{(x_B + 1)^2 + (3 - x_B - 4)^2} = 3\sqrt{2}$$

$$2(x_B + 1)^2 = 18$$

$$x_B + 1 = \pm 3$$

So either $x_B = 2$ and then $B = (2, 1)$ or $x_B = -4$ and $B = (-4, 7)$.

Exercise 3.2.9a The line segment $\overline{AB}$ has endpoint $A(2, 1)$. Find the coordinates of B, given that the gradient of this line segment is 2 and the length $\overline{AB}$ is $\sqrt{5}$.

Exercise 3.2.9b The line segment $\overline{AB}$ has endpoint $A(0, 3)$. Find the coordinates of B, given that the gradient of this line segment is -2 and the length $\overline{AB}$ is $2\sqrt{5}$.

Exercise 3.2.9c The line segment $\overline{AB}$ has endpoint $A(-2, 0)$. Find the coordinates of B, given that the gradient of this line segment is $\frac{1}{2}$ and the coordinates of points B satisfy the equation $y = x$.

Exercise 3.2.9d The line segment $\overline{AB}$ has endpoint $A(-3, -1)$. Find the coordinates of B, given that the gradient of this line segment is 2 and the coordinates of points B satisfy the equation $x + y = 2$.

Exercise 3.2.9e The line segment $\overline{AB}$ has endpoint $A(-1, 1)$. Find the coordinates of B, given that the gradient of this line segment is 1 and the coordinates of points B satisfy the equation $y = x^2$.

Equation of a circle

1. Find the distance between $(3, 1)$ and $(6, 5)$.

2. Write down an expression for the distance between $(3, 1)$ and (x, y).

A circle with centre at C is a set of all points at a given distance from C.

3. Show that the equation for the circle centered at $(3, 1)$ of radius 5 can be written as:

$$(x-3)^2 + (y-1)^2 = 5^2$$

4. Find the equation of a circle with the given centre and radius:

(a) $(1, 2), \quad r = 3$

(b) $(0, -3), \quad r = 1$

(c) $(-3, -2), \quad r = \sqrt{5}$

(b) $(-1, 5), \quad r = \frac{1}{2}$

5. Find the centre and the radius of the circles given by the following equations:

(a) $(x-2)^2 + (y-7)^2 = 6^2$

(b) $x^2 + y^2 = 9$

(c) $(x+1)^2 + (y-4)^2 = 2$

(d) $x^2 + (y-5)^2 = 7$

6. Find the centre and the radius of the circles given by the following equations:

(a) $x^2 - 2x + 1 + y^2 - 4x + 4 = 25$

(b) $x^2 + 2x + y^2 - 6x = 5$

(c) $x^2 + 4x + y^2 - 8y + 19 = 0$

(d) $x^2 + x + y^2 - 3y = 0$

SHORT TEST

1. [3 *points*]

For the line segment $\overline{AB}$ with endpoints $A(-1,-4)$ and $B(3,2)$, find:

(a) midpoint, (b) length, (c) gradient.

2. [5 *points*]

Let $A(-2,0), B(1,2)$ and $C(c,3)$. Find c, if:

(a) A, B and C are collinear.

(b) The midpoint of AC has coordinates $(4,1.5)$.

(c) $|AB| = |AC|$.

3. [4 *points*]

The line segment $\overline{AB}$ has endpoint $A(1,-2)$. Find the coordinates of B, given that the gradient of this line segment is 3 and the length $\overline{AB}$ is $\sqrt{10}$.

SHORT TEST
SOLUTIONS

1. [3 *points*]
For the line segment $\overline{AB}$ with endpoints $A(-1,-4)$ and $B(3,2)$, find:

(a) midpoint, (b) length, (c) gradient.

(a) $M_{AB} = \left(\frac{-1+3}{2}, \frac{-4+2}{2}\right) = (1,-1)$.

(b) $|AB| = \sqrt{(3-(-1))^2 + (2-(-4))^2} = 2\sqrt{13}$

(c) $m_{AB} = \frac{2-(-4)}{3-(-1)} = \frac{3}{2}$

2. [5 *points*]
Let $A(-2,0), B(1,2)$ and $C(c,3)$. Find c, if:

(a) A, B and C are collinear.

(b) The midpoint of AC has coordinates $(4, 1.5)$.

(c) $|AB| = |AC|$.

(a) $m_{AB} = \frac{2}{3}$, so $\frac{2}{3} = \frac{3}{c+2}$, which gives $c = 2.5$.

(b) $\frac{-2+c}{2} = 4$, so $c = 10$

(c) $|AB| = \sqrt{3^2+2^2} = \sqrt{13}$, so

$$\sqrt{13} = \sqrt{(c+2)^2 + 9}$$

which gives $(c+2)^2 = 4$, so $c + 2 = \pm 2$ and so $c = 0$ or $c = -4$.

3. [4 *points*]
The line segment $\overline{AB}$ has endpoint $A(1,-2)$. Find the coordinates of B, given that the gradient of this line segment is 3 and the length $\overline{AB}$ is $\sqrt{10}$.

We have $\Delta x = x_B - 1$ and $\Delta y = y_B + 2$. Using the gradient we have:

$$\frac{y_B + 2}{x_B - 1} = 3$$

which gives $y_B = 3x_B - 5$. Now using the length and subsituting the above into the length formula:

$$\sqrt{(x_B-1)^2 + (3x_B-3)^2} = \sqrt{10}$$

This gives $(x_B - 1)^2 = \pm 1$, so $x_B = 2$ or $x_B = 0$ and hence B has either coordinates $(2,1)$ or $(0,-5)$.

1.3 Equation of a line

An equation of the form:

$$Ax + By = C$$

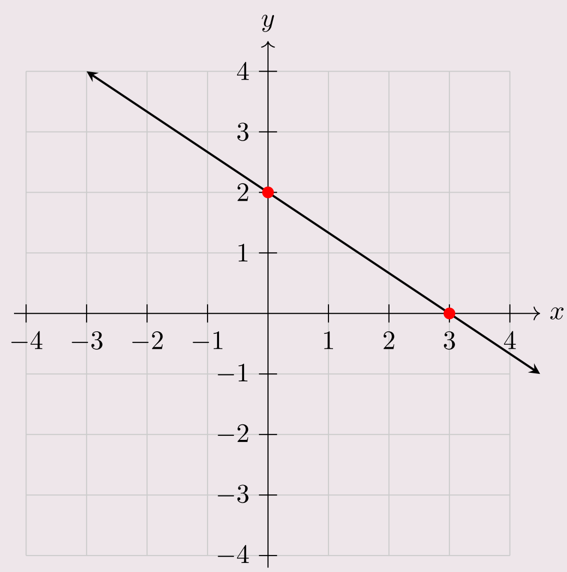
is called a linear equation and the set of points satisfying this equation forms a line on a 2d coordinate plane. This form of an equation of a line is called **general form**.

Worked example 3.3.1.

Sketch the line represented by the equation:

$$2x + 3y = 6$$

In order to sketch a line we need two points. We will find the points of intersection of the line with the axes. If $x = 0$, then $3y = 6$, so $y = 2$, which means that the line intersects the y-axis at $(0, 2)$. If $y = 0$, then $2x = 6$, so $x = 3$ and the line intersects the x-axis at $(3, 0)$. This is enought to sketch the line:



Note that we find the y-intercept by substitution $x = 0$, since every point on the y-axis has x-coordinate equal to 0. Similarly, we find the x-intercept by substituting $y = 0$, since every point on the x-axis has y-coordinate equal to 0.

Special cases arise if $A = 0$ or $B = 0$. If $A = 0$ (and $B \neq 0$), then the equation becomes $By = C$ and it represents a horizontal line. Similarly, if $B = 0$ (and $A \neq 0$), then the equation becomes $Ax = C$ and it represents a vertical line. Finally if both A and B are 0, then the equation becomes $0 = C$, which is satisfied by every pairs (x, y), if $C = 0$, and by no pair, if $C \neq 0$.

Exercise 3.3.1. Sketch the lines represented by the following equations:

(a) $3x - 4y = 12$

(b) $2x + y = 6$

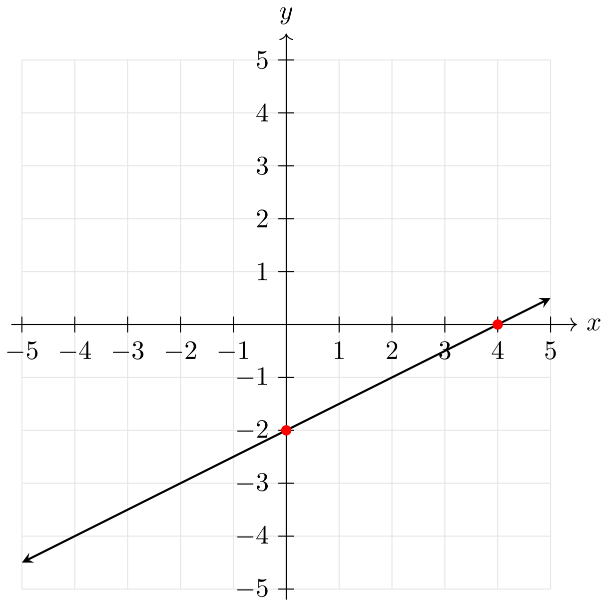
(c) $x - 3y = 3$

(d) $\frac{1}{2}x + \frac{1}{3}y = 1$

(e) $\sqrt{2}x - y = \sqrt{2}$

(f) $x - 4y = -2$

Consider the following line:



It has an x-intercept at $(4, 0)$ and y-intercept at $(0, -2)$. We can use these points to find an equation of this line. We substitute these points into the equation:

$$Ax + By = C$$

to get $4A = C$ and $-2B = C$, we can then express every unknown in terms of one of them (for example in terms of A). We get $C = 4A$ and $B = -2A$, so the equation is:

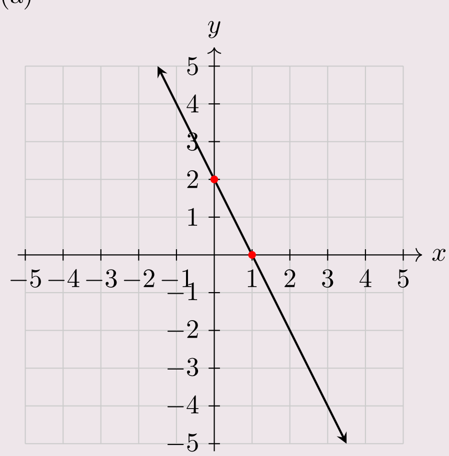
$$Ax - 2Ay = 4A$$

dividing both sides by A we get:

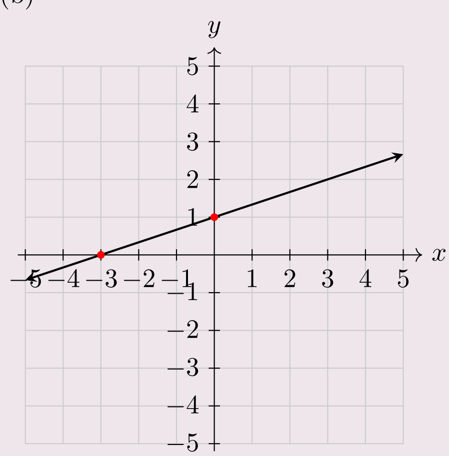
$$x - 2y = 4$$

Exercise 3.3.2. Find the equations of the following lines in the form $Ax + By = C$:

(a)



(b)



Another way of thinking of a line is in terms of its gradient. The gradient of a line is the gradient of any line segment contained in this line. Suppose that we would like to find a line which passes through the point $(1, 2)$ and has a gradient of 2. Any point (x, y) on this line must satisfy the equation:

$$\frac{y-2}{x-1} = 2$$

which can be rearranged into:

$$y - 2 = 2(x - 1)$$

In general an equation of a line with gradient m and passing through (x_0, y_0) is:

$$y - y_0 = m(x - x_0)$$

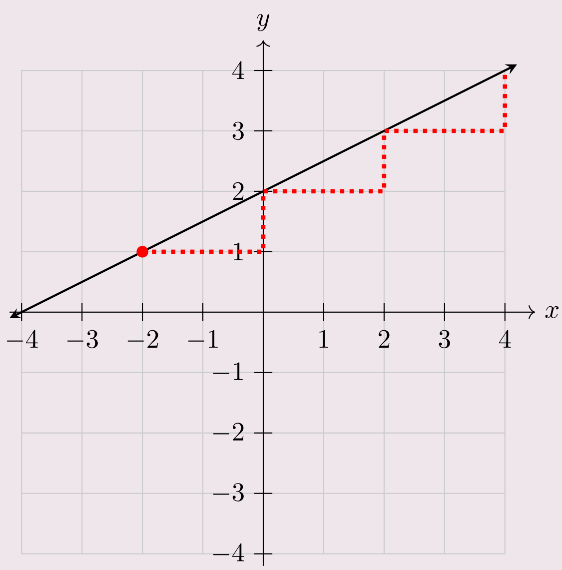
This form of an equation is known as **point - gradient** (or point-slope) form.

Worked example 3.3.3.

Sketch the line represented by the equation:

$$y - 1 = \frac{1}{2}(x + 2)$$

From the form of the equation we can see that it must have a gradient of $\frac{1}{2}$ and it must pass through $(-2, 1)$. We can use this information to sketch the line:



Note that the dotted lines represent the gradient. The gradient of $\frac{1}{2}$ indicates that whenever x-coordinate is increased by 2 ($\Delta x = 2$), then y-coordinate must increase by 1 ($\Delta y = 1$).

Exercise 3.3.3. Sketch the lines represented by the following equations:

(a) $y - 1 = 2(x - 2)$

(b) $y - 3 = \frac{1}{2}(x + 1)$

(c) $y - 3 = x + 2$

(d) $y + 1 = 2x$

(e) $y = -\frac{1}{3}(x - 3)$

(f) $y + 2 = -\frac{2}{3}(x + 3)$

Worked example 3.3.4.

Find a point-gradient equation of the line that passes through the points $(-2, 3)$ and $(1, -6)$.

We need to find the gradient of the line using the two points. We have $\Delta x = 1 - (-2) = 3$ and $\Delta y = -6 - 3 = -9$

$$m = \frac{-9}{3} = -3$$

The point-gradient equation is then:

$$y - 3 = -3(x + 2)$$

Note that we could have also written the equation:

$$y + 6 = -3(x - 1)$$

Make sure to convince yourself that the two above equations are equivalent.

Exercise 3.3.4. Find a point-gradient equation for the lines satisfying the following coniditions.

(a) Gradient $= \frac{1}{2}$ and passes through $(2, 4)$.

(b) Gradient $= -\frac{2}{3}$ and passes through $(-1, 5)$.

(c) Passes through $(1, 1)$ and $(3, -7)$.

(d) Gradient $= -3$ and passes through $(3, -2)$.

(e) Passes through $(-2, 0)$ and $(0, 4)$.

(f) Gradient $= -1$ and intersects the x-axis at $x = 5$.

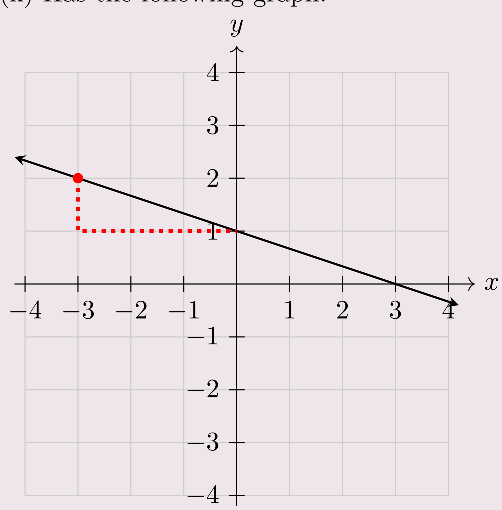
(g) Passes through $(-3, -1)$ and $(2, 2)$.

(h) Gradient $= 5$ and intersects the y-axis at $y = -1$.

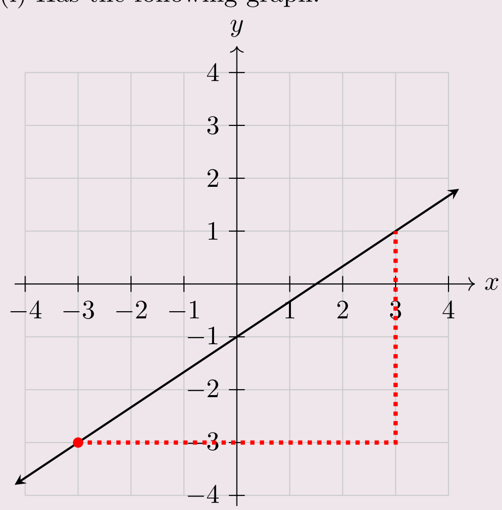
(i) Passes through $(2, -2)$ and $(4, -2)$.

(j) Gradient $= 0$ and passes through $(1, 3)$.

(k) Has the following graph:



(l) Has the following graph:



If a line has gradient m and intersects the y-axis (y-intercept) at $y = c$, then its equation in the point-gradient form will be:

$$y - c = mx$$

which rearranges into

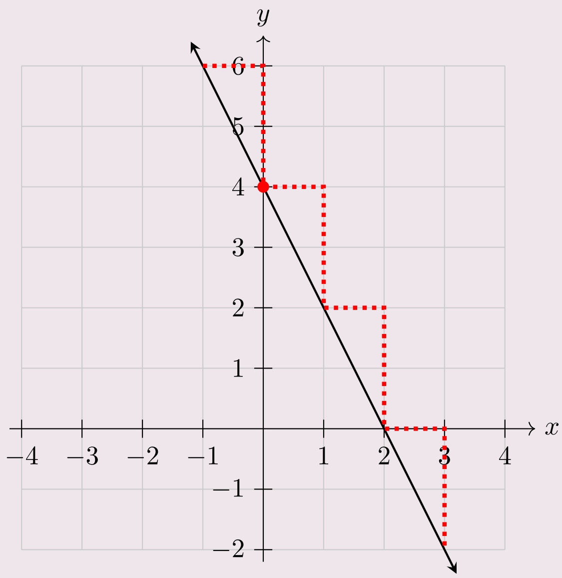
$$y = mx + c$$

This is known as the **gradient-intercept** form.

Worked example 3.3.4.

Sketch the line given by the equation $y = -2x + 4$

The equation is in the gradient-intercept form, the gradient is -2 and the y-intercept is 4 (the function intersects the y-axis at $y = 4$). Since the gradient is -2 an increase in x coordinate by 1 ($\Delta x = 1$) corresponds to a decrease in y-coordinate by 2 ($\Delta y = -2$).



Note that it is useful to find the x-intercept before sketching the line. At the x-axis the y-coordinate is 0, so we need to solve the equation:

$$0 = -2x + 4$$

which gives $x = 2$, so the line intersects the x-axis at $x = 2$.

Exercise 3.3.4. Sketch the lines represented by the following equations:

(a) $y = -x + 3$

(b) $y = \frac{1}{2}x + 1$

(c) $y = -\frac{1}{3}x - 2$

(d) $y = 2x - 6$

(e) $y = \sqrt{2}x - 3\sqrt{2}$

(f) $y = 3$

Worked example 3.3.5.

Find the gradient-intercept equation of the line passing through the points $(-1, 1)$ and $(3, -1)$.

Method 1
We can substitute the coordinates of the points into the equation $y = mx + c$. This gives us two equations:

$$\begin{cases} 1 = -m + c \\ -1 = 3m + c \end{cases}$$

Solving the system gives $m = -\frac{1}{2}$ and $c = \frac{1}{2}$, so the equation is:

$$y = -\frac{1}{2}x + \frac{1}{2}$$

Method 2
We calculate the gradient first. We have $\Delta x = 3 - (-1) = 4$ and $\Delta y = -1 - 1 = -2$, so:

$$m = \frac{-2}{4} = -\frac{1}{2}$$

We can now proceed in two ways:

Method 2a We write down the equation in the point-gradient form and rearrange it:

$$y - 1 = -\frac{1}{2}(x + 1)$$

$$y = -\frac{1}{2}x - \frac{1}{2} + 1$$

$$y = -\frac{1}{2}x + \frac{1}{2}$$

Method 2b Substitute coordinates of one of the points to find c:

$$1 = -\frac{1}{2} \times (-1) + c$$

which gives $c = \frac{1}{2}$, so:

$$y = -\frac{1}{2}x + \frac{1}{2}$$

Exercise 3.3.5. Find the gradient-intercept equation of the line passing through the following pairs of points:

(a) $(-1, 2), \quad (1, -4)$

(b) $(-3, 1), \quad (1, 2)$

(c) $(0, 2), \quad (6, 0)$

(d) $(\sqrt{2}, 1), \quad (2, \sqrt{2})$

(e) $(1, 1), \quad (4, 1)$

(f)$(-1, 3), \quad (-1, 5)$

Worked example 3.3.6.

Consider the line given by the equation:

$$2x - 3y = 12$$

(a) Find axis intercepts of the line.

(b) Find the gradient of the line.

(c) Decide if the points: $(4, -3)$, $(3, -2)$, $(-2, 5)$ lie on the line.

(d) Sketch the line.

(a) To find y-intercept we substitute $x = 0$:

$$-3y = 12$$

which gives $y = -4$. To find x-intercept we substitute $y = 0$:

$$2x = 12$$

which gives $x = 6$. So the axis intercepts are $(0, -4)$ and $(6, 0)$.

(b) We can rearrange the equation into the gradient-intercept form:

$$2x - 12 = 3y$$

$$\frac{2}{3}x - 4 = y$$

so the gradient is $m = \frac{2}{3}$.

(c) Substituting the points into the equation:

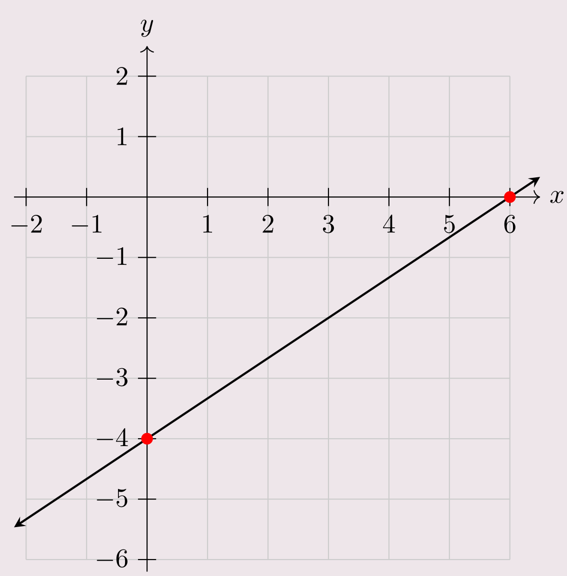
$$2 \times 4 - 3 \times (-3) \neq 12$$

$$2 \times 3 - 3 \times (-2) = 12$$

$$2 \times (-2) - 3 \times 5 \neq 12$$

so the second point lies on the line, the other two do not.

(d)



Exercise 3.3.6. For the lines given by the following equations (i) find the axis intercepts, (ii) find the gradient, (iii) decide if points A, B and C lie on the line and (iv) sketch the lines.

(a) $3x - y = 6, \quad A(1,-3),\ B(2,2),\ C(3,3)$

(b) $x + 3y = 3, \quad A(1,1),\ B(3,-1),\ C(6,-1)$

(c) $2x + 5y = 10, \quad A(1,2),\ B(2,1),\ C(-2,3)$

(d) $4x - 3y = 24, \quad A(3,-4),\ B(9,4),\ C(1.5,-6)$

A **point of intersection** of two lines must satisfy the equations of both lines. In order to find coordinates of such point we solve a system of simultaneous equations.

Worked example 3.3.7.

Find the coordinates of the point of intersection of the lines $x + 2y = 4$ and $y = \frac{1}{4}x + \frac{1}{2}$ and sketch both lines.

We solve the system:

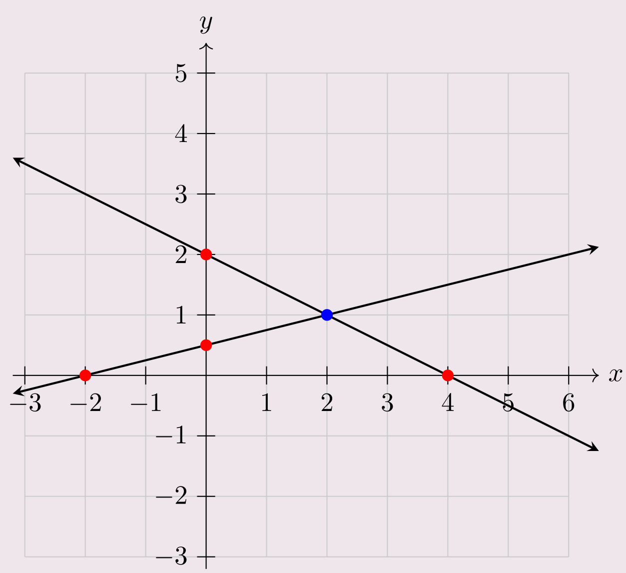
$$\begin{cases} x + 2y = 4 \\ y = \dfrac{1}{4}x + \dfrac{1}{2} \end{cases}$$

We can substitute the value of y from the second equation into the first one to get:

$$x + 2\left(\frac{1}{4}x + \frac{1}{2}\right) = 4$$

Solving this gives $x = 2$ and if $x = 2$, then we get $y = 1$. So the point of intersection has coordinates $(2, 1)$.

We sketch the lines by finding the axis intercepts. Line $x + 2y = 4$ has axis intercepts at $(4, 0)$ and $(0, 2)$. Line $y = \dfrac{1}{4}x + \dfrac{1}{2}$ has axis intercepts at $(-2, 0)$ and $(0, \frac{1}{2})$. We get the following graph:



Exercise 3.3.7. Find the coordinates of the point of intersection of the following lines and sketch these lines.

(a) $l_1 : y = 2x - 3, \quad l_2 : 3x - 2y = 3$

(b) $l_1 : 3x + y = 4, \quad l_2 : 2x - 3y = -1$

(c) $l_1 : y = \frac{1}{2}x + 3, \quad l_2 : x + 3y = 14$

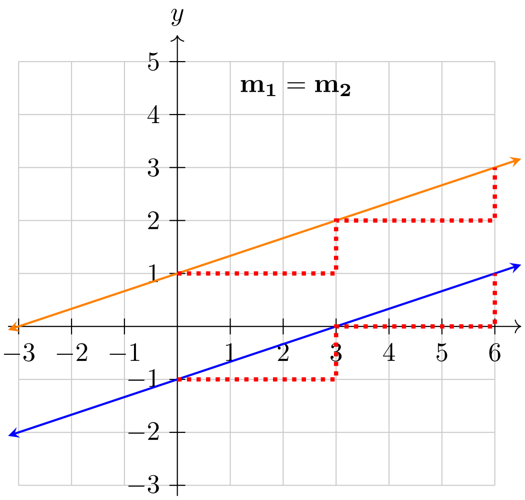
(d) $l_1 : y = -2, \quad l_2 : y = \frac{2}{3}x - 1$

(e) $l_1 : y = \sqrt{2}x - 1, \quad l_2 : x = \sqrt{2}$

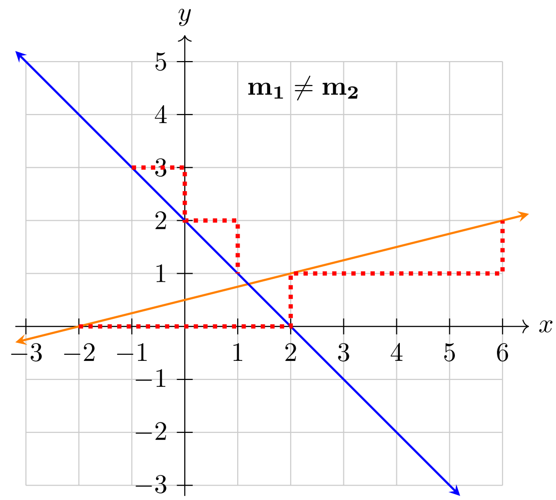
(f) $l_1 : 2x + 3y = -3, \quad l_2 : y = -\frac{2}{3}x + 2$

In part (f) of the above exercise we've encountered lines that do not intersect. These lines are parallel. Two lines in 2d plane are parallel, if they do not intersect. In particular that means that they must have the same gradient and do not coincide.

Parallel lines



Intersecting lines



When two lines l_1 and l_2 are parallel we write $l_1 \parallel l_2$.

Worked example 3.3.8.

Find the value of A for which the following lines are parallel:

$$l_1 : Ax + 3y = 7$$

$$l_2 : y = \frac{3}{4}x - 1$$

The gradient of the second line is $\frac{3}{4}$, in order to find the gradient of the first line, we rearrange it to the gradient-intercept form:

$$y = -\frac{A}{3}x + \frac{7}{3}$$

So the gradient of the first line is $-\frac{A}{3}$. We have $m_1 = -\frac{A}{3}$ and $m_2 = \frac{3}{4}$. For the lines to be parallel we must have $m_1 = m_2$, so:

$$-\frac{A}{3} = \frac{3}{4}$$

which gives $A = -\frac{9}{4}$. Note that the lines do not coincide as they have different y-intercepts.

Exercise 3.3.8a Find the value(s) of the parameter k for which the given lines are parallel.

(a) $l_1: y = 4x + 1, \quad l_2: y = kx + 2$

(b) $l_1: 2x + y = 4, \quad l_2: y = kx - 1$

(c) $l_1: y = \frac{2}{3}x + 3, \quad l_2: kx - 3y = 1$

(d) $l_1: 3x - 4y = 2, \quad l_2: kx - 6y = 1$

(e) $l_1: y = 5, \quad l_2: kx + 3y = 4$

(f) $l_1: x = 3, \quad l_2: 2x - ky = 1$

(g) $l_1: kx - y = 4, \quad l_2: 4x + ky = 1$

(h) $l_1: y = (k+1)x - 3, \quad l_2: y = (k^2 - 1)x + 1$

Exercise 3.3.8b Find an equation of the line parallel to line l and passing through point A.

(a) $l: y = 3x + 1, \quad A(1, 1)$

(b) $l: y = \frac{1}{2}x + 2, \quad A(6, -1)$

(c) $l: 3x + 2y = 5, \quad A(2, -1)$

(d) $l: x - 3y = 1, \quad A(-1, 1)$

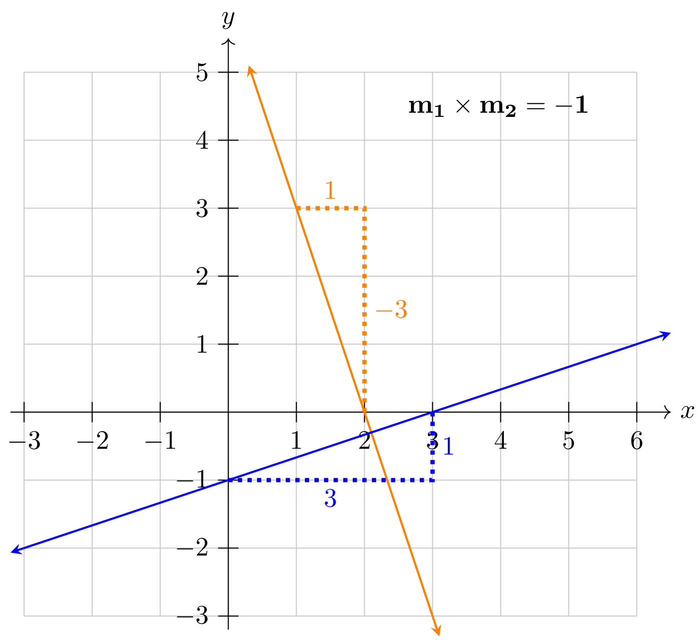
(e) $l: y = 5, \quad A(2, 4)$

(f) $l: x = 1, \quad A(3, -1)$

Two lines are perpendicular, if they intersect at a right angle. Lines are perpendicular, if the product of their gradients is -1 (or, in other words, one gradient is the negative reciprocal of the other):

$$m_1 \times m_2 = -1$$

$$m_2 = -\frac{1}{m_1}$$



Note that we use the following notation to denote the fact that two lines l_1 and l_2 are perpendicular: $l_1 \perp l_2$. A line is perpendicular to a line segment, if its perpendicular to the line containing this line segment. Similarly two line segments are perpendicular two each other, if the lines containing these segements are perpendicular.

Worked example 3.3.9.

Find the equation of and sketch the line which is perpendicular to $2x + 3y = 1$ and passes through $(4, 3)$.

Rearranging $2x + 3y = 1$ we get

$$y = -\frac{2}{3}x + \frac{1}{3}$$

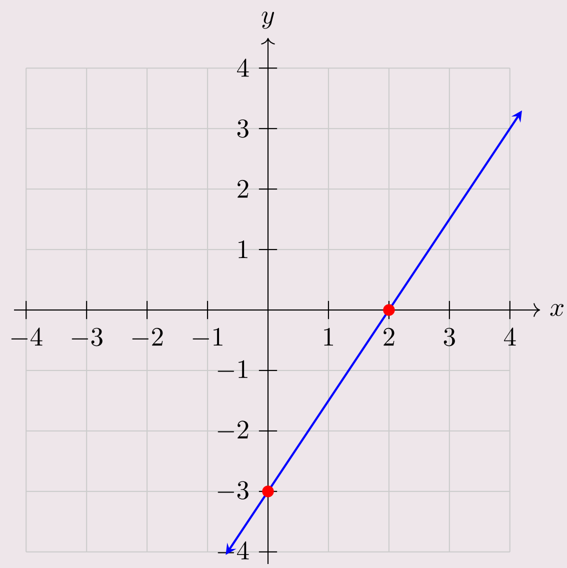
so it has a gradient of $-\frac{2}{3}$. The gradient of the perpendicular line will then be $\frac{3}{2}$. It has to pass through $(4, 3)$, so its equation in point-gradient form is:

$$y - 3 = \frac{3}{2}(x - 4)$$

This can be rearranged in the general form:

$$3x - 2y = 6$$

This line has x-intercept at $(2, 0)$ and y-intercept at $(0, -3)$, so its graph is as follows:



Exercise 3.3.9a

Find the value(s) of the parameter q for which the given lines are perpendicular.

(a) $l_1 : y = 2x + 1, \quad l_2 : y = qx - 3$

(b) $l_1 : 3x - y = 4, \quad l_2 : y = qx + 2$

(c) $l_1 : y = \frac{3}{4}x + 1, \quad l_2 : qx - 2y = 3$

(d) $l_1 : 2x - 3y = 1, \quad l_2 : qx - 4y = 5$

(e) $l_1 : y = 5, \quad l_2 : 3x + qy = 4$

(f) $l_1 : x = 3, \quad l_2 : qx - y = 5$

(g) $l_1 : qx - 2y = 3, \quad l_2 : 8x + qy = 7$

(h) $l_1 : y = (2q - 3)x - 3, \quad l_2 : y = (q - 3)x + 1$

Exercise 3.3.9b Find an equation, in general form with integer coefficients, of the line perpendicular to line l and passing through point A.

(a) $l: y = 3x + 1, \quad A(3, 5)$

(b) $l: y = -\frac{1}{4}x + 1, \quad A(1, -2)$

(c) $l: 2x + 3y = 5, \quad A(1, -1)$

(d) $l: x - 2y = 1, \quad A(2, 1)$

(e) $l: y = 1, \quad A(3, -1)$

(f) $l: x = 5, \quad A(4, -2)$

Worked example 3.3.10.

Find the equation, in general form, of the line containing the height from vertex C of the triangle with vertices $A(-1, 2)$, $B(3, 0)$ and $C(1, 6)$.

We start by finding the gradient of the line segment $\overline{AB}$:

$$m_{AB} = \frac{0 - 2}{3 - (-1)} = -\frac{1}{2}$$

The line containing the height from vertex C will be perpendicular to this line segment, so it will have gradient equal to 2. Its equation in point-gradient form will then be:

$$y - 6 = 2(x - 1)$$

Rearranging to the general form we get:

$$2x - y = -4$$

Exercise 3.3.10a Find the equation of the line containing the height from vertex C of the triangle ABC, where:

(a) $A(-3, 0),\ B(1, 1),\ C(0, 4)$

(b) $A(-3, 0),\ B(1, 1),\ C(0, 4)$

(c) $A(-3, 0),\ B(1, 1),\ C(0, 4)$

Exercise 3.3.10b Consider a triangle ABC with $A(-2, -3),\ B(4, 1)$ and $C(-1, 3)$.

(a) Find the equation of the line passing through A and B.

(b) Find the equation of the line containing height from vertex C.

(c) Find the point of intersection of the lines found in part (a) and (b).

(d) Find the area of the triangle ABC.

Exercise 3.3.10c Find the area of a triangle with vertices at $A(-4, 1),\ B(3, 0)$ and $C(4, 3)$.

Exercise 3.3.10d Consider a triangle ABC with $A(-3, -1),\ B(1, 1)$ and $C(0, 5)$.

(a) Find the equations of lines containing heights from vertex A and B and find the point of intersection of these lines.

(b) Show that the point found in (a) lies on the line containing height from vertex C.

Worked example 3.3.11.

Show that the triangle with vertices at $A(-2,2), B(2,1)$ and $C(-1,6)$ is a right triangle and find its area.

We find the gradients of all the line segments:

$$m_{AB} = \frac{1-2}{2-(-2)} = -\frac{1}{4}$$

$$m_{BC} = \frac{6-1}{-1-2} = -\frac{5}{3}$$

$$m_{AC} = \frac{6-2}{-1-(-2)} = 4$$

We have $m_{AB} \times m_{AC} = -1$, so the line segments $\overline{AB}$ and $\overline{AC}$ are perpendicular and the triangle is a right triangle with right angle at vertex A. Now we have:

$$|AB| = \sqrt{(-1)^2 + 4^2} = \sqrt{17}$$

$$|AC| = \sqrt{4^2 + 1^2} = \sqrt{17}$$

which shows that the triangle is also isosceles. Its area is:

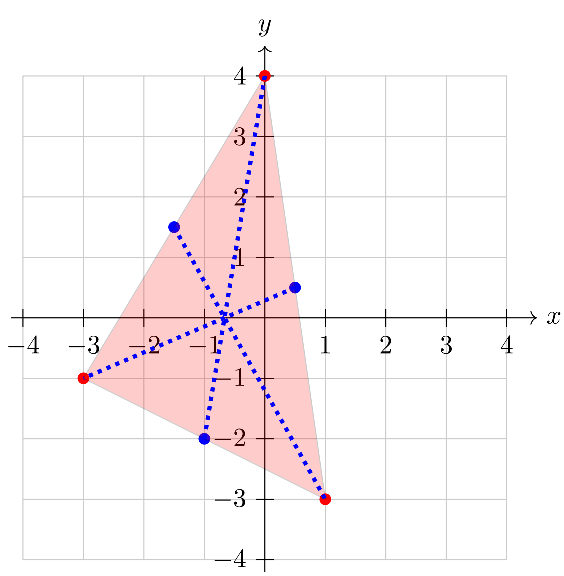
$$Area = \frac{|AB| \times |AC|}{2} = \frac{17}{2}$$

Exercise 3.3.11

In the following cases decide, if the triangle ABC is a right triangle. Justify your answer.

(a) $A(-5,0),\ B(3,-1),\ C(-4,4)$ (b) $A(1,1),\ B(3,4),\ C(4,-1)$ (c) $A(5,5),\ B(2,3),\ C(0,-1)$

A **median** of a triangle is a line segment joining a vertex to the midpoint of the opposite side.



Exercise 3.3.12a
Find the equation of the line containing the median from vertex C for the triangle ABC with:

(a) $A(0,0),\ B(6,2),\ C(4,10)$ (b) $A(-1,-2),\ B(3,4),\ C(5,0)$ (c) $A(-1,-3),\ B(3,5),\ C(1,4)$

Exercise 3.3.12b
Consider a triangle ABC with $A(-4,2),\ B(2,0),\ C(-2,6)$.

(a) Find the equations of the lines containing medians from the vertices A and B.

(b) Find the point of intersection between the lines found in part (a).

(c) Show that the point you found in part (b) lies on the line containing the median from vertex C.

Worked example 3.3.13.

Find the coordinates of any point on the line $2x+y=3$ at a distance of $\sqrt{17}$ from the point $(3,3)$.

If $A(x_A, y_A)$ lies on the line $2x+y=3$, then we must have:

$$2x_A + y_A = 3$$

and hence $y_A = 3-2x_A$
If the distance form A to $(3,3)$ is $\sqrt{17}$, then we must also have:

$$\sqrt{(x_A-3)^2+(y_A-3)^2} = \sqrt{17}$$

Substituting the $y_A = 3-2x_A$ into the above equation we get:

$$\sqrt{(x_A-3)^2+(-2x_A)^2} = \sqrt{17}$$

$$5x_A^2 - 6x_A + 9 = 17$$

$$(5x_A+4)(x_A-2) = 0$$

which gives $x_A = 2$ or $x_A = -\frac{4}{5}$, so the points are $(2,-1)$ and $(-\frac{4}{5}, \frac{23}{5})$.

Exercise 3.3.13
Find the coordinates of any point on the line l at a distance d from the point P.

(a) $l: x-2y=1,\ d=\sqrt{5},\ P(1,2)$

(b) $l: 2x-y=6,\ d=5,\ P(0,5)$

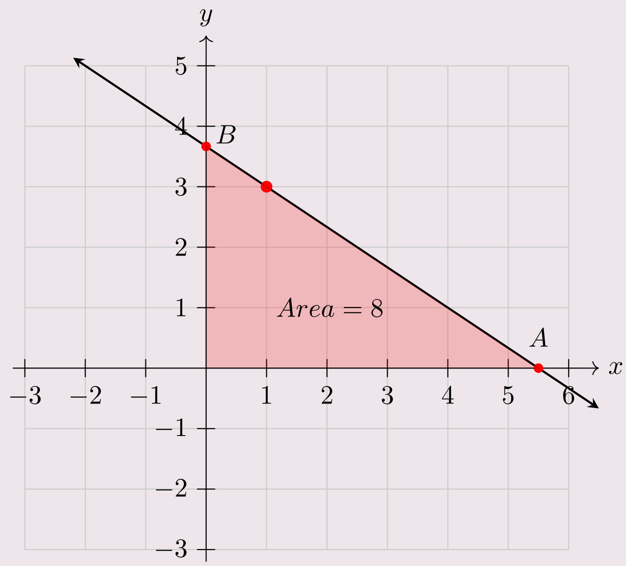
(c) $l: 3x+y=4,\ d=\sqrt{37},\ P(-1,-2)$

(d) $l: x+4y=1,\ d=\sqrt{26},\ P(2,0)$

Worked example 3.3.14.

Find an equation of the line passing through $(1, 3)$, which intersects the positive x-axis at A and positive y-axis at B, so that the triangle AOB has an area equal to 8.

The line cannot be vertical, so we will work with gradient-intercept form of the equation.



We start with the equation $y = mx + c$. The y-intercept is $(0, c)$, the x-intercept is $(-\frac{c}{m}, 0)$. Since the line passes through $(1, 3)$, then we must have:

$$3 = m + c$$

so $m = 3 - c$. Using the area of the triangle AOB we get:

$$\frac{c \times (-\frac{c}{3-c})}{2} = 8$$

$$-c^2 = 48 - 16c$$

$$0 = (c - 4)(c - 12)$$

$c = 4$ or $c = 12$ and the two possible lines are $y = -x + 4$ and $y = -9x + 12$.

Exercise 3.3.14a Find an equation of the line passing through $(3, 2)$, which intersects the positive x-axis at A and positive y-axis at B, so that the triangle AOB has an area equal to 16.

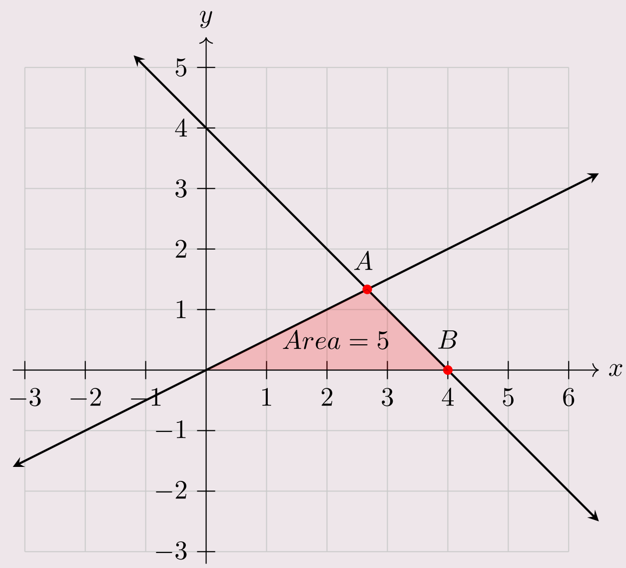
Exercise 3.3.14b Find an equation of the line passing through $(-2, 1)$, which intersects the negative x-axis at A and positive y-axis at B, so that the triangle AOB has an area equal to $5\frac{1}{3}$.

Exercise 3.3.14c Find an equation of the line passing through $(-5, -1)$, which intersects the negative x-axis at A and negative y-axis at B, so that the triangle AOB has an area equal to 10.

Worked example 3.3.15.

Let l be a line with equation $y = 4 - x$. Find an equation of line k, which passes through the origin and intersects the line l in point A located in the first quadrant, such that the area of the triangle AOB is equal to 5, where B is the x-intercept of line l.

The line l intersects the x-axis at $(4, 0)$. If k was vertical, then since it passes through the origin, it would have the equation $x = 0$, it would then intersect the line l at $(0, 4)$ and the area of the triangle would be 8. So k is not vertical. Let $y = mx+c$ be the equation for k. It passes through the origin, so $c = 0$.



We need to find the coordinates of point A - the intersection of the two lines. We solve the system:

$$\begin{cases} y = 4 - x \\ y = mx \end{cases}$$

We get $x = \dfrac{4}{m+1}$, $\quad y = \dfrac{4m}{m+1}$. Now the base of the triangle has length of 4 and the height of the triangle is the y-coordinate of point A, so it is $frac4mm + 1$. Since the area of the triangle is 5 we get that:

$$\frac{4 \times \frac{4m}{m+1}}{2} = 5$$

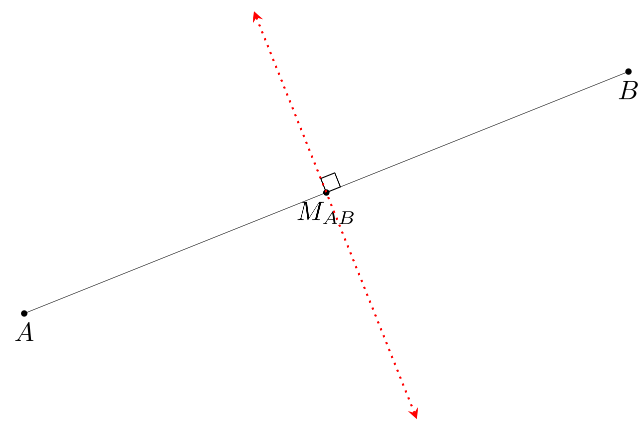
Solving this we get $m = \frac{5}{3}$, so line k has the equation $y = \frac{5}{3}x$.

Note that in the example above point A was in the first quadrant. This was stipulated in the question. If this restriction was not present what would change?

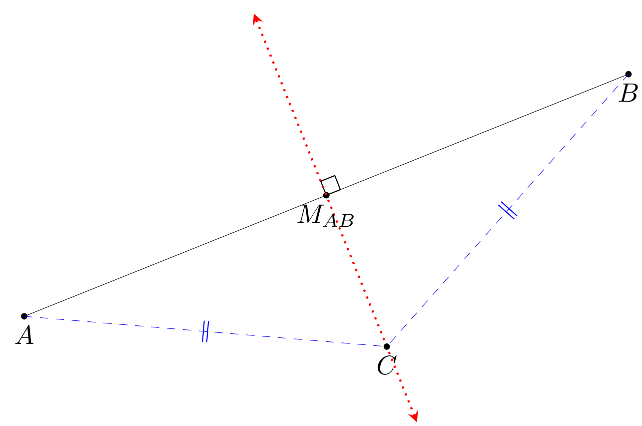
Exercise 3.3.15a Solve the worked example 3.3.14 for the case in which A lies in the IV quadrant.

Exercise 3.3.15b Suppose line l is given by $x - 2y = 4$. Determine the equation of line k, which passes through the origin and intersects line l at point A, such that the area of triangle AOB equals 6, where B is the y-intercept of line l.

A **perpendicular bisector** of a line segment $\overline{AB}$ is a line perpendicular to $\overline{AB}$ that passes through the midpoint of $\overline{AB}$.



Note that a perpendicular bisector of a line segment $\overline{AB}$ is also a set of points **equidistant** from the endpoints of the line segment.



Exercise 3.3.16a
Find the equation of the perpendicular bisector of the line segment $\overline{AB}$ for the following endpoints.

(a) $A(-2,1),\ B(6,3)$ (b) $A(-1,6),\ B(1,0)$ (c) $A(-5,-3),\ B(0,2)$

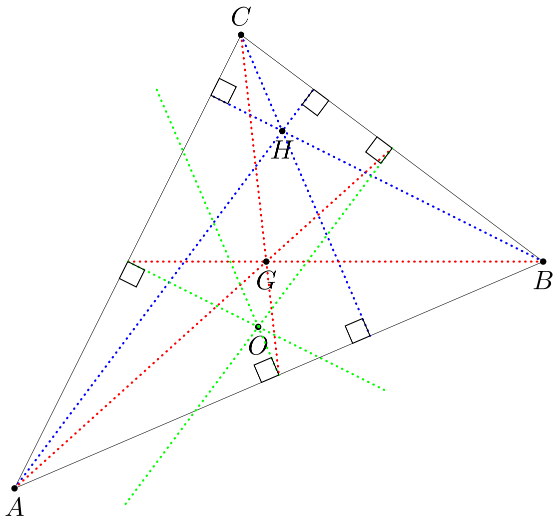
Exercise 3.3.16b
Consider a triangle ABC with $A(-3,1),\ B(1,-5)$ and $C(-1,3)$.

(a) Find the perpendicular bisectors of the line segments $\overline{AB}$ and $\overline{BC}$.

(b) Find the point of intersection of the lines found in part (a).

(c) Explain why the point found in (b) must also lie on the perpendicular bisector of the line segment $\overline{AC}$.

The point of intersection of the heights of a triangle is called the **orthocenter**, the point of intersection of the medians is called the **centroid**, the point of intersection of the perpendicular bisectors of the sides is called the **circumcenter** (it is the center of the circle circumscribing the triangle).



On the diagram above the blue dotted lines represent the heights and H is the orthocenter, the red dotted lines represent the medians and G is the centroid and the green dotted lines represented the perpendicular bisectors of the sides and O is the circumcenter.

Exercise 3.3.17
Consider a triangle ABC with $A(-5,3)$, $B(3,-1)$ and $C(-1,5)$.

(a) Find the coordinates of the orthocenter.

(b) Find the coordinates of the centroid.

(c) Find the coordinates of the circumcenter.

(d) Show that the orthocenter, the centroid and the circumcenter lie on one line and find its equation.

The line you found in part (d) of the above exercise is known as the **Euler line** after the mathematician Leonhard Euler.

Parabola

Consider the line $l : y = 0$ (the x-axis) and the point $P(0, 2)$.

1. Consider point $A(2, 2)$. Show that the distance from A to l is the same as the distance from A to P.

2. Now consider point $B(4, 5)$. Show that the distance from B to l is also the same as the distance from B to P.

3. Let $C(x, y)$ be a point that is equidistant from l and P. Show that

$$y = \frac{1}{4}x^2 + 1$$

4. Hence sketch the set of points that are equidistant from line l and point P.

5. Find an equation for the set of points that are equidistant from line $y = -1$ and $(0, 3)$ and sketch these points.

6. Find an equation for the set of points that are equidistant from line $y = p$ and $(0, q)$.

7. Find an equation for the set of points that are equidistant from line $x = 1$ and $(0, 2)$ and sketch these points.

SHORT TEST

1. [6 *points*]

Find the equation, in the gradient-intercept form, of the line satisfying the following conditions.

(a) Passes through $(-1, 3)$ and $(3, 1)$.

(b) Passes through the origin and is parallel to $3x + 2y = 4$.

(c) Passes through $(-2, 2)$ and is perpendicular to $y - 1 = 2(x - 3)$.

2. [9 *points*]

Consider a triangle ABC with $A(-4, 2), B(6, -3)$ and $C(0, 4)$.

(a) Find the equation line containing the height from vertex C.

(b) Find the coordinates of the point of intersection of the line found in (a) with the line containing the line segment $\overline{AB}$.

(c) Find the perpendicular bisector of the line segment $\overline{AC}$.

(d) Find the equation of the line containing the median from vertex B.

SHORT TEST
SOLUTIONS

1. [6 *points*]
Find the equation, in the gradient-intercept form, of the line satisfying the following conditions.

(a) Passes through $(-1, 3)$ and $(3, 1)$.

(b) Passes through the origin and is parallel to $3x + 2y = 4$.

(c) Passes through $(-2, 2)$ and is perpendicular to $y - 1 = 2(x - 3)$.

(a) $m = \frac{1-3}{3-(-1)} = -\frac{1}{2}$, so the equation is $y = -\frac{1}{2}x + c$, substituting one of the points we get:

$$3 = -\frac{1}{2} \times (-1) + c$$

so $c = \frac{5}{2}$, so the equation is $y = -\frac{1}{2}x + \frac{5}{2}$.

(b) Rearranging $3x + 2y = 4$ into gradient-intercept form we get $y = -\frac{3}{2}x + 2$. The gradient is $-\frac{3}{2}$, a parallel line will have the same gradient and since it passes through the origing the y-intercept is 0. So the equation is $y = -\frac{3}{2}x$.

(c) The gradient of $y - 1 = 2(x - 3)$ is 2, so a perpendicular line will have gradient of $-\frac{1}{2}$. The equation is then $y = -\frac{1}{2}x + c$. Substituting the point $(-2, 2)$ we get:

$$2 = -\frac{1}{2} \times (-2) + c$$

which gives $c = 1$, so the equation is $y = -\frac{1}{2}x + 1$

2. [9 *points*]
Consider a triangle ABC with $A(-4, 2), B(6, -3)$ and $C(0, 4)$.

(a) Find the equation line containing the height from vertex C.

The gradient of $\overline{AB}$ is $m_{AB} = \frac{-3-2}{6-(-4)} = -\frac{1}{2}$, so the gradient of the line containing the height will be 2. It must pass through C, so the y-intercept is 4 and the equation is:

$$y = 2x + 4$$

(b) Find the coordinates of the point of intersection of the line found in (a) with the line containing the line segment $\overline{AB}$.

The line containing $\overline{AB}$ has gradient $-\frac{1}{2}$ (found in part (a)). Substituting one of the endpoints we get:

$$2 = -\frac{1}{2} \times (-4) + c$$

so $c = 0$ and the equation is $y = -\frac{1}{2}x$.

SHORT TEST

SOLUTIONS

We need to solve:

$$\begin{cases} y = 2x + 4 \\ y = -\frac{1}{2}x \end{cases}$$

which gives $x = -1.6$ and $y = 0.8$, so the point of intersection is $(-1.6, 0.8)$

(c) Find the perpendicular bisector of the line segment $\overline{AC}$.

The gradient of $\overline{AC}$:

$$m_{AC} = \frac{4-2}{0-(-4)} = \frac{1}{2}$$

So the gradient of the perpendicular bisector will be -2. Midpoint of $\overline{AC}$:

$$M_{AC} = \left(\frac{-4+0}{2}, \frac{2+4}{2}\right) = (-2, 3)$$

The equation of the perpendicular bisector (in point-gradient form) is then:

$$y - 3 = -2(x + 3)$$

(d) Find the equation of the line containing the median from vertex B.

Midpoint of $\overline{AC}$ is $(-2, 3)$ (found in part (c)), the gradient of the line is then:

$$m_{BM_{AC}} = \frac{3-(-3)}{-2-6} = -\frac{3}{4}$$

Given that it must pass through B, the equation in point-gradient form is:

$$y + 3 = -\frac{3}{4}(x - 6)$$