Complex numbers [58 marks]

1. [Maximum mark: 18] SPM.1.AHL.TZ0.11

(a) Express $-3+\sqrt{3}\mathrm{i}$ in the form $r\mathrm{e}^{\mathrm{i}\theta}$, where $r>0$ and $-\pi<\theta\leqslant\pi$. [5]

Markscheme

attempt to find modulus ***(M1)***

$r=2\sqrt{3}\left(=\sqrt{12}\right)$ ***A1***

attempt to find argument in the correct quadrant ***(M1)***

$\theta=\pi+\arctan\left(-\frac{\sqrt{3}}{3}\right)$ ***A1***

$=\frac{5\pi}{6}$ ***A1***

$-3+\sqrt{3}\mathrm{i}=\sqrt{12}\mathrm{e}^{\frac{5\pi\mathrm{i}}{6}}\left(=2\sqrt{3}\mathrm{e}^{\frac{5\pi\mathrm{i}}{6}}\right)$

[5 marks]

Let the roots of the equation $z^3=-3+\sqrt{3}\mathrm{i}$ be u, v and w.

(b) Find u, v and w expressing your answers in the form $r\mathrm{e}^{\mathrm{i}\theta}$, where $r>0$ and $-\pi<\theta\leqslant\pi$. [5]

Markscheme

attempt to find a root using de Moivre's theorem ***M1***

$12^{\frac{1}{6}}\mathrm{e}^{\frac{5\pi\mathrm{i}}{18}}$ ***A1***

attempt to find further two roots by adding and subtracting $\frac{2\pi}{3}$ to the argument ***M1***

$12^{\frac{1}{6}}\mathrm{e}^{-\frac{7\pi\mathrm{i}}{18}}$ ***A1***

$12^{\frac{1}{6}}\mathrm{e}^{\frac{17\pi\mathrm{i}}{18}}$ ***A1***

Note: Ignore labels for u, v and w at this stage.

[5 marks]

On an Argand diagram, u, v and w are represented by the points U, V and W respectively.

(c) Find the area of triangle UVW. [4]

Markscheme

METHOD 1

attempting to find the total area of (congruent) triangles UOV, VOW and UOW ***M1***

Area $= 3\left(\frac{1}{2}\right)\left(12^{\frac{1}{6}}\right)\left(12^{\frac{1}{6}}\right)\sin\frac{2\pi}{3}$ ***A1A1***

Note: Award ***A1*** for $\left(12^{\frac{1}{6}}\right)\left(12^{\frac{1}{6}}\right)$ and ***A1*** for $\sin\frac{2\pi}{3}$

$= \frac{3\sqrt{3}}{4}\left(12^{\frac{1}{3}}\right)$ (or equivalent) ***A1***

METHOD 2

$\text{UV}^2 = \left(12^{\frac{1}{6}}\right)^2 + \left(12^{\frac{1}{6}}\right)^2 - 2\left(12^{\frac{1}{6}}\right)\left(12^{\frac{1}{6}}\right)\cos\frac{2\pi}{3}$ (or equivalent) ***A1***

$\text{UV} = \sqrt{3}\left(12^{\frac{1}{6}}\right)$ (or equivalent) ***A1***

attempting to find the area of UVW using Area = $\frac{1}{2}$ × UV × VW × sin α for example ***M1***

Area $= \frac{1}{2}\left(\sqrt{3}\times 12^{\frac{1}{6}}\right)\left(\sqrt{3}\times 12^{\frac{1}{6}}\right)\sin\frac{\pi}{3}$

$= \frac{3\sqrt{3}}{4}\left(12^{\frac{1}{3}}\right)$ (or equivalent) ***A1***

[4 marks]

(d) By considering the sum of the roots u, v and w, show that

$$\cos\frac{5\pi}{18} + \cos\frac{7\pi}{18} + \cos\frac{17\pi}{18} = 0.$$

[4]

Markscheme

$u + v + w = 0$ ***R1***

$12^{\frac{1}{6}}\left(\cos\left(-\frac{7\pi}{18}\right) + \text{i}\sin\left(-\frac{7\pi}{18}\right) + \cos\frac{5\pi}{18} + \text{i}\sin\frac{5\pi}{18} + \cos\frac{17\pi}{18} + \text{i}\sin\frac{17\pi}{18}\right) = 0$ ***A1***

consideration of real parts ***M1***

$12^{\frac{1}{6}}\left(\cos\left(-\frac{7\pi}{18}\right) + \cos\frac{5\pi}{18} + \cos\frac{17\pi}{18}\right) = 0$

$\cos\left(-\frac{7\pi}{18}\right) = \cos\frac{17\pi}{18}$ explicitly stated ***A1***

$\cos\frac{5\pi}{18} + \cos\frac{7\pi}{18} + \cos\frac{17\pi}{18} = 0$ ***AG***

[4 marks]

2. [Maximum mark: 20] EXN.1.AHL.TZ0.12

(a) Use the binomial theorem to expand $(\cos\theta + \text{i}\sin\theta)^4$. Give your answer in the form $a + b\text{i}$ where a and b are expressed in terms of $\sin\theta$ and $\cos\theta$. [3]

Markscheme

* This sample question was produced by experienced DP mathematics senior examiners to aid teachers in preparing for external assessment in the new MAA course. There may be minor differences in formatting compared to formal exam papers.

uses the binomial theorem on $(\cos\theta + \mathrm{i}\sin\theta)^4$ **M1**

$= {}_4C_0\cos^4\theta + {}_4C_1\cos^3\theta(\mathrm{i}\sin\theta) + {}_4C_2\cos^2\theta(\mathrm{i}^2\sin^2\theta) + {}_4C_3\cos\theta(\mathrm{i}^3\sin^3\theta) + {}_4C_4(\mathrm{i}^4\sin^4\theta)$
A1

$= (\cos^4\theta - 6\cos^2\theta\sin^2\theta + \sin^4\theta) + \mathrm{i}(4\cos^3\theta\sin\theta - 4\cos\theta\sin^3\theta)$ **A1**

[3 marks]

(b) Use de Moivre's theorem and the result from part (a) to show that $\cot 4\theta = \frac{\cot^4\theta - 6\cot^2\theta + 1}{4\cot^3\theta - 4\cot\theta}$. [5]

Markscheme

(using de Moivre's theorem with $n = 4$ gives) $\cos 4\theta + \mathrm{i}\sin 4\theta$ **(A1)**

equates both the real and imaginary parts of $\cos 4\theta + \mathrm{i}\sin 4\theta$ and
$(\cos^4\theta - 6\cos^2\theta\sin^2\theta + \sin^4\theta) + \mathrm{i}(4\cos^3\theta\sin\theta - 4\cos\theta\sin^3\theta)$ **M1**

$\cos 4\theta = \cos^4\theta - 6\cos^2\theta\sin^2\theta + \sin^4\theta$ and $\sin 4\theta = 4\cos^3\theta\sin\theta - 4\cos\theta\sin^3\theta$

recognizes that $\cot 4\theta = \frac{\cos 4\theta}{\sin 4\theta}$ **(A1)**

substitutes for $\sin 4\theta$ and $\cos 4\theta$ into $\frac{\cos 4\theta}{\sin 4\theta}$ **M1**

$\cot 4\theta = \frac{\cos^4\theta - 6\cos^2\theta\sin^2\theta + \sin^4\theta}{4\cos^3\theta\sin\theta - 4\cos\theta\sin^3\theta}$

divides the numerator and denominator by $\sin^4\theta$ to obtain

$\cot 4\theta = \frac{\frac{\cos^4\theta - 6\cos^2\theta\sin^2\theta + \sin^4\theta}{\sin^4\theta}}{\frac{4\cos^3\theta\sin\theta - 4\cos\theta\sin^3\theta}{\sin^4\theta}}$ **A1**

$\cot 4\theta = \frac{\cot^4\theta - 6\cot^2\theta + 1}{4\cot^3\theta - 4\cot\theta}$ **AG**

[5 marks]

(c) Use the identity from part (b) to show that the quadratic equation $x^2 - 6x + 1 = 0$ has roots $\cot^2\frac{\pi}{8}$ and $\cot^2\frac{3\pi}{8}$. [5]

Markscheme

setting $\cot 4\theta = 0$ and putting $x = \cot^2\theta$ in the numerator of $\cot 4\theta = \frac{\cot^4\theta - 6\cot^2\theta + 1}{4\cot^3\theta - 4\cot\theta}$ gives $x^2 - 6x + 1 = 0$
M1

attempts to solve $\cot 4\theta = 0$ for θ **M1**

$4\theta = \frac{\pi}{2}, \frac{3\pi}{2}, \ldots \ \left(4\theta = \frac{1}{2}(2n+1)\pi, \mathrm{n} = 0, 1, \ldots\right)$ **(A1)**

$\theta = \frac{\pi}{8}, \frac{3\pi}{8}$ **A1**

Note: Do not award the final **A1** if solutions other than $\theta = \frac{\pi}{8}, \frac{3\pi}{8}$ are listed.

finding the roots of $\cot 4\theta = 0$ $\left(\theta = \frac{\pi}{8}, \frac{3\pi}{8}\right)$ corresponds to finding the roots of $x^2 - 6x + 1 = 0$ where $x = \cot^2 \theta$ **R1**

so the equation $x^2 - 6x + 1 = 0$ as roots $\cot^2 \frac{\pi}{8}$ and $\cot^2 \frac{3\pi}{8}$ **AG**

[5 marks]

(d) Hence find the exact value of $\cot^2 \frac{3\pi}{8}$. [4]

Markscheme

attempts to solve $x^2 - 6x + 1 = 0$ for x **M1**

$x = 3 \pm 2\sqrt{2}$ **A1**

since $\cot^2 \frac{\pi}{8} > \cot^2 \frac{3\pi}{8}$, $\cot^2 \frac{3\pi}{8}$ has the smaller value of the two roots **R1**

Note: Award **R1** for an alternative convincing valid reason.

so $\cot^2 \frac{3\pi}{8} = 3$ -2 $\sqrt{2}$ **A1**

[4 marks]

(e) Deduce a quadratic equation with integer coefficients, having roots $\operatorname{cosec}^2 \frac{\pi}{8}$ and $\operatorname{cosec}^2 \frac{3\pi}{8}$. [3]

Markscheme

let $y = \operatorname{cosec}^2 \theta$

uses $\cot^2 \theta = \operatorname{cosec}^2 \theta - 1$ where $x = \cot^2 \theta$ **(M1)**

$x^2 - 6x + 1 = 0 \Rightarrow (y-1)^2 - 6(y-1) + 1 = 0$ **M1**

$y^2 - 8y + 8 = 0$ **A1**

[3 marks]

3. [Maximum mark: 20] 24N.1.AHL.TZ0.12

Consider the equation $z^4 = 16\mathrm{i}$, where $z \in \mathbb{C}$.

The equation has four roots z_1, z_2, z_3, z_4, where $z_i = r(\cos\theta_i + \mathrm{i}\sin\theta_i), r > 0$ and $0 \le \theta_1 < \theta_2 < \theta_3 < \theta_4 < 2\pi$.

(a) Find z_1, z_2, z_3 and z_4. [6]

Markscheme

$|16\mathrm{i}| = 16$ and $\arg(16\mathrm{i}) = \frac{\pi}{2}$ ***(A1)***

attempt to use De Moivre's Theorem ***(M1)***

$z_1 = 2\left(\cos\left(\frac{\pi}{8}\right) + \mathrm{i}\sin\left(\frac{\pi}{8}\right)\right)$ ***A1***

attempts to find other solutions using $z = 2\left(\cos\left(\frac{\pi}{8} + \frac{\pi k}{2}\right) + \mathrm{i}\sin\left(\frac{\pi}{8} + \frac{\pi k}{2}\right)\right)$ or equivalent ***(M1)***

$z_2 = 2\left(\cos\left(\frac{5\pi}{8}\right) + \mathrm{i}\sin\left(\frac{5\pi}{8}\right)\right)$ (or any other root) ***A1***

$z_3 = 2\left(\cos\left(\frac{9\pi}{8}\right) + \mathrm{i}\sin\left(\frac{9\pi}{8}\right)\right)$ and $z_4 = 2\left(\cos\left(\frac{13\pi}{8}\right) + \mathrm{i}\sin\left(\frac{13\pi}{8}\right)\right)$ ***A1***

Note: Award a maximum of ***(A1)(M1)A1(M1)A1A0*** for more than four roots or any roots outside the range.

Note: Allow use of r-cis form throughout.

[6 marks]

The roots z_1, z_2, z_3 and z_4 form a geometric sequence.

(b) Find the common ratio of the sequence, expressing your answer in Cartesian form. [3]

Markscheme

attempt to evaluate a ratio with their roots eg $\frac{z_2}{z_1}$ ***(M1)***

$\frac{z_2}{z_1} = \frac{2\left(\cos\left(\frac{5\pi}{8}\right)+\mathrm{i}\sin\left(\frac{5\pi}{8}\right)\right)}{2\left(\cos\left(\frac{\pi}{8}\right)+\mathrm{i}\sin\left(\frac{\pi}{8}\right)\right)}$ or equivalent

$= \left(\cos\left(\frac{\pi}{2}\right) + \mathrm{i}\sin\left(\frac{\pi}{2}\right)\right)$ ***(A1)***

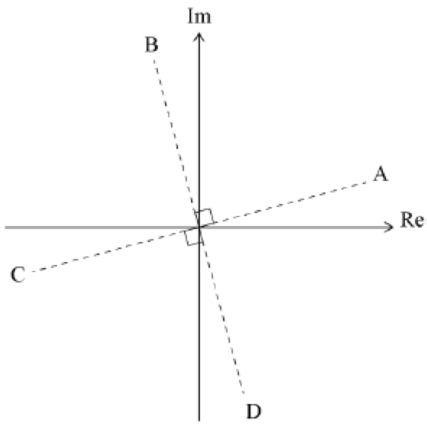
$= \mathrm{i}$ ***A1***

[3 marks]

The roots z_1, z_2, z_3 and z_4 are represented by the points $\mathrm{A}, \mathrm{B}, \mathrm{C}$ and D respectively on an Argand diagram.

(c) Plot the points $\mathrm{A}, \mathrm{B}, \mathrm{C}$ and D on an Argand diagram. [3]

Markscheme



point A in approximately correct place in first quadrant ***A1***

points $\mathrm{A}, \mathrm{B}, \mathrm{C}$ and D approximately the same distance from the origin ***A1***

approximate angular separation of $\frac{\pi}{2}$ ***A1***

Note: Dotted lines not required.

[3 marks]

The equation $v^4 = a + b\mathrm{i}$, where $v \in \mathbb{C}$ and $a, b \in \mathbb{R}$ has roots z_1^*, z_2^*, z_3^* and z_4^*.

(d) Determine the value of a and the value of b. [3]

Markscheme

EITHER

$\left(z_i^*\right)^4 = \left(z_1^4\right)^*$ ***(A1)***

$= (16\mathrm{i})^*$ ***(A1)***

OR

$z_1^* = 2\left(\cos\left(-\frac{\pi}{8}\right) + \mathrm{i}\sin\left(-\frac{\pi}{8}\right)\right)$ ***(A1)***

$\left(z_1^*\right)^4 = 2^4\left(\cos\left(-\frac{4\pi}{8}\right) + \mathrm{i}\sin\left(-\frac{4\pi}{8}\right)\right)$ ***(A1)***

OR

$z_1 z_2 z_3 z_4 = -16\mathrm{i}$ ***(A1)***

$(z_1 z_2 z_3 z_4)^* = (-16\mathrm{i})^*$

$z_1^* z_2^* z_3^* z_4^* = 16\mathrm{i}$ ***(A1)***

THEN

$(z^4 =) - 16\mathrm{i}$ ***A1***

$(a = 0,\ b = -16)$

[3 marks]

The midpoint of $[\mathrm{AB}]$ is A', the midpoint of $[\mathrm{BC}]$ is B', the midpoint of $[\mathrm{CD}]$ is C' and the midpoint of $[\mathrm{DA}]$ is D'.

Consider the equation $w^p = 2^q$, where $w \in \mathbb{C}$ and $p, q \in \mathbb{Z}^+$.

Four of the roots of $w^p = 2^q$ are represented by the points A', B', C' and D'.

(e) Find the least possible value of p and the corresponding value of q. [5]

Markscheme

$\arg w_1 \ \left(= \frac{\pi}{8} + \frac{\pi}{8}\right) \ = \frac{3\pi}{8}$ ***A1***

$\mathrm{AB} = \sqrt{2^2 + 2^2} = \sqrt{8} = 2\sqrt{2}$

$\Rightarrow \mathrm{AA}' = \sqrt{2}$

$\Rightarrow \mathrm{OA}' = |w_1| = \sqrt{2}$ ***A1***

$\therefore w_1 = \sqrt{2}\ \mathrm{cis}\ \frac{3\pi}{8}$

considers when $\arg\left({w_1}^p\right) \in \mathbb{Z}^+$ (multiple of 2π) ***(M1)***

$\Rightarrow p = 16$ ***A1***

$\Rightarrow q = 8$ ***A1***

[5 marks]