Complex numbers [58 marks]

1. [Maximum mark: 18] SPM.1.AHL.TZ0.11

(a) Express $-3+\sqrt{3}\mathrm{i}$ in the form $r\mathrm{e}^{\mathrm{i}\theta}$, where $r>0$ and $-\pi<\theta\leqslant\pi$. [5]

Let the roots of the equation $z^3=-3+\sqrt{3}\mathrm{i}$ be u, v and w.

(b) Find u, v and w expressing your answers in the form $r\mathrm{e}^{\mathrm{i}\theta}$, where $r>0$ and $-\pi<\theta\leqslant\pi$. [5]

On an Argand diagram, u, v and w are represented by the points U, V and W respectively.

(c) Find the area of triangle UVW. [4]

(d) By considering the sum of the roots u, v and w, show that

$$\cos\frac{5\pi}{18}+\cos\frac{7\pi}{18}+\cos\frac{17\pi}{18}=0.$$ [4]

2. [Maximum mark: 20] EXN.1.AHL.TZ0.12

(a) Use the binomial theorem to expand $(\cos\theta+\mathrm{i}\sin\theta)^4$. Give your answer in the form $a+b\mathrm{i}$ where a and b are expressed in terms of $\sin\theta$ and $\cos\theta$. [3]

(b) Use de Moivre's theorem and the result from part (a) to show that $\cot 4\theta=\frac{\cot^4\theta-6\cot^2\theta+1}{4\cot^3\theta-4\cot\theta}$. [5]

(c) Use the identity from part (b) to show that the quadratic equation $x^2-6x+1=0$ has roots $\cot^2\frac{\pi}{8}$ and $\cot^2\frac{3\pi}{8}$. [5]

(d) Hence find the exact value of $\cot^2\frac{3\pi}{8}$. [4]

(e) Deduce a quadratic equation with integer coefficients, having roots $\operatorname{cosec}^2 \frac{\pi}{8}$ and $\operatorname{cosec}^2 \frac{3\pi}{8}$. [3]

3. [Maximum mark: 20] 24N.1.AHL.TZ0.12

Consider the equation $z^4 = 16\mathrm{i}$, where $z \in \mathbb{C}$.

The equation has four roots z_1, z_2, z_3, z_4, where $z_i = r\left(\cos\theta_i + \mathrm{i}\sin\theta_i\right)$, $r > 0$ and $0 \le \theta_1 < \theta_2 < \theta_3 < \theta_4 < 2\pi$.

(a) Find z_1, z_2, z_3 and z_4. [6]

The roots z_1, z_2, z_3 and z_4 form a geometric sequence.

(b) Find the common ratio of the sequence, expressing your answer in Cartesian form. [3]

The roots z_1, z_2, z_3 and z_4 are represented by the points $\mathrm{A}, \mathrm{B}, \mathrm{C}$ and D respectively on an Argand diagram.

(c) Plot the points $\mathrm{A}, \mathrm{B}, \mathrm{C}$ and D on an Argand diagram. [3]

The equation $v^4 = a + b\mathrm{i}$, where $v \in \mathbb{C}$ and $a, b \in \mathbb{R}$ has roots z_1^*, z_2^*, z_3^* and z_4^*.

(d) Determine the value of a and the value of b. [3]

The midpoint of $[\mathrm{AB}]$ is A', the midpoint of $[\mathrm{BC}]$ is B', the midpoint of $[\mathrm{CD}]$ is C' and the midpoint of $[\mathrm{DA}]$ is D'.

Consider the equation $w^p = 2^q$, where $w \in \mathbb{C}$ and $p, q \in \mathbb{Z}^+$.

Four of the roots of $w^p = 2^q$ are represented by the points $\mathrm{A}', \mathrm{B}', \mathrm{C}'$ and D'.

(e) Find the least possible value of p and the corresponding value of q. [5]