1.5 Transformations

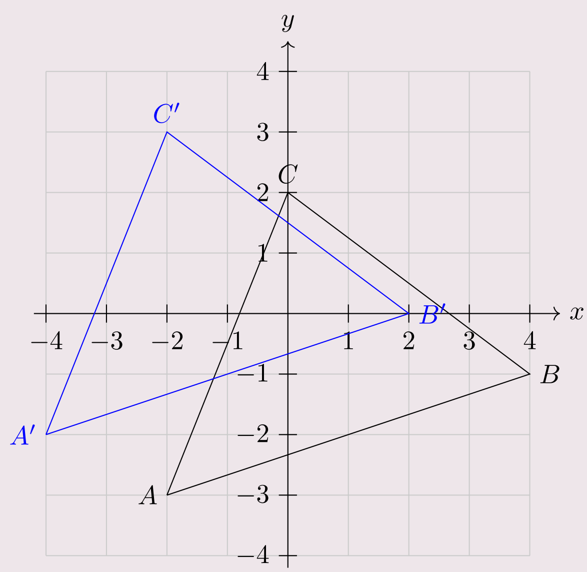
In this section we will look at transformations of a 2-dimensional plane. The transformations we will look at are:

- translations - when every point is shifted be a vector $\vec{v}$.
- reflections - when every point is reflected in line l. In particular we will look at reflections in the horizontal and vertical lines (including reflections in x and y axes) and the line $y = x$.
- rotations - when every point is rotated by an angle around a fixed centre of rotation. We will look at rotation around the origin by multiples of 90°.
- dilations - we will consider dilations parallel to axes when the distance from every point to the horizontal or vertical axis is scaled by a given factor.

Worked example 3.5.1.

Find the image of the triangle with vertices at $A(-2,-3), B(4,-1)$ and $C(0,2)$ after translation by a vector $\vec{v} = \begin{pmatrix} -2 \\ 1 \end{pmatrix}$.

Translating by vector $\vec{v} = \begin{pmatrix} -2 \\ 1 \end{pmatrix}$ corresponds to decreasing the first coordinate by 2 and increasing the second coordinate by 1. The image of the triangle ABC will be a triangle $A'B'C'$ with $A' = (-4,-2), B' = (2,0)$ and $C' = (-2,3)$.



Exercise 3.5.1a Find the image of the triangle with vertices $A(-3,1), B(2,-2)$ and $C(1,4)$ after translation by the following vectors. Draw both triangles.

(a) $\vec{v} = \begin{pmatrix} 1 \\ 3 \end{pmatrix}$ (b) $\vec{v} = \begin{pmatrix} -1 \\ 2 \end{pmatrix}$ (c) $\vec{v} = \begin{pmatrix} -2 \\ -3 \end{pmatrix}$ (d) $\vec{v} = \begin{pmatrix} 3 \\ -4 \end{pmatrix}$

Exercise 3.5.1b Find the image of the parallelogram with vertices $A(-4,-3), B(0,-2), C(1,1)$ and $D(-3,0)$ after translation by the following vectors. Draw both triangles.

(a) $\vec{v} = \begin{pmatrix} 2 \\ 0 \end{pmatrix}$ (b) $\vec{v} = \begin{pmatrix} 0 \\ -1 \end{pmatrix}$ (c) $\vec{v} = \begin{pmatrix} 3 \\ 2 \end{pmatrix}$ (d) $\vec{v} = \begin{pmatrix} 1 \\ -1 \end{pmatrix}$

Worked example 3.5.2.

The line given by the equation $y = \frac{1}{3}x + 1$ is translated by the vector $\vec{v} = \begin{pmatrix} 3 \\ -4 \end{pmatrix}$. Find the equation of the resulting line and sketch both lines.

We have

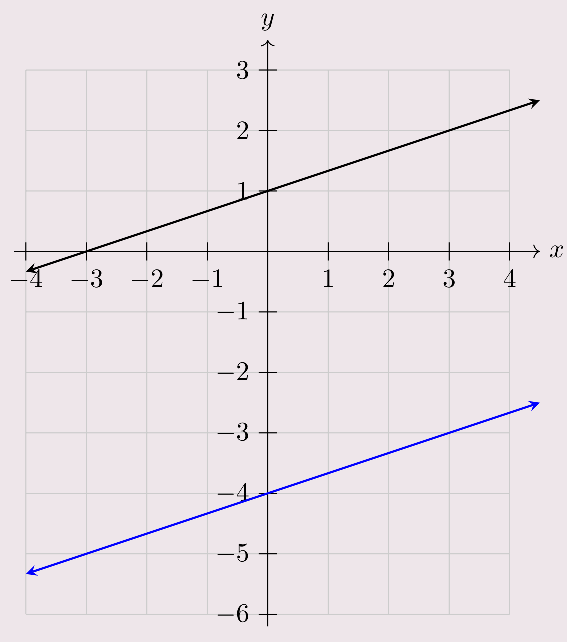
$$\begin{cases} x' = x + 3 \\ y' = y - 4 \end{cases}$$

which gives $x = x' - 3$ and $y = y' + 4$, so the equation of the image is:

$$y' + 4 = \frac{1}{3}(x' - 3) + 1$$

which gives $y' = \dfrac{1}{3}x' - 4$.

The diagram below shows both lines (black before and blue after translation).



Exercise 3.5.2. Line l is translated by the vector $\vec{v}$. Find the equation of the resulting line and sketch both lines.

(a) $l : y = 2x - 4 \qquad \vec{v} = \begin{pmatrix} 0 \\ 2 \end{pmatrix}$

(b) $l : y = -\frac{1}{2}x + 1 \qquad \vec{v} = \begin{pmatrix} 4 \\ 0 \end{pmatrix}$

(c) $l : y = 3 - x \qquad \vec{v} = \begin{pmatrix} 1 \\ -2 \end{pmatrix}$

(d) $l : 2x + 3y = 3 \qquad \vec{v} = \begin{pmatrix} -1 \\ 1 \end{pmatrix}$

Worked example 3.5.3.

The curve given by the equation $y = x^2$ is translated by the vector $\vec{v} = \begin{pmatrix} 3 \\ 1 \end{pmatrix}$. Find the equation of the resulting curve and sketch both curves.

We have

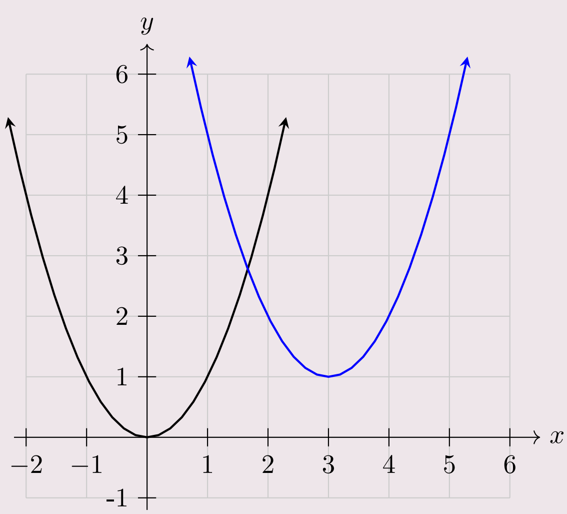
$$\begin{cases} x' = x + 3 \\ y' = y + 1 \end{cases}$$

which gives $x = x' - 3$ and $y = y' - 1$, so the equation of the image is:

$$y' - 1 = (x' - 3)^2$$

which gives $y' = (x' + 3)^2 + 1$.

The diagram below shows both curves (black before and blue after translation).



Exercise 3.5.3. Curve C is translated by the vector $\vec{v}$. Find the equation of the resulting curve and sketch both curves

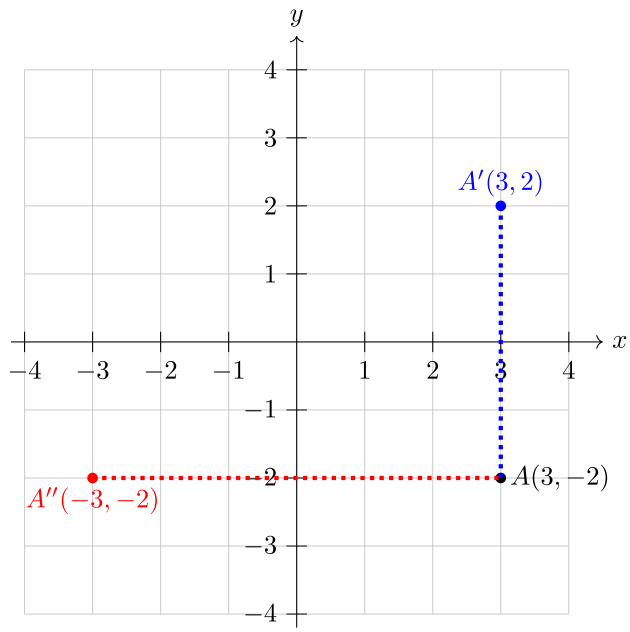
(a) $C : y = x^2, \quad \vec{v} = \begin{pmatrix} -1 \\ -2 \end{pmatrix}$

(b) $C : xy = 1, \quad \vec{v} = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$

(c) $C : y = 2^x, \quad \vec{v} = \begin{pmatrix} 0 \\ -2 \end{pmatrix}$

Reflecting any point in the x-axis results in the change of the sign of the y-coordinate of that point (the x-coordinate remains unchanged). Similarly reflecting in the y-axis results in the change of the sign of the x-coordinate (y-coordinate is unchanged).

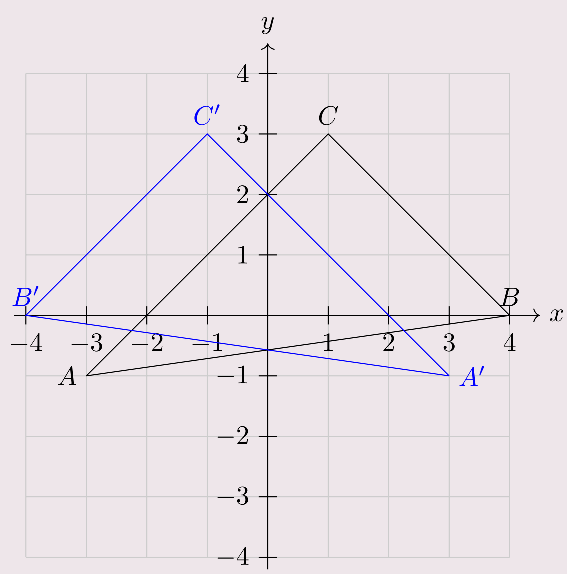
The diagram below shows point $A(3,-2)$ reflected in the x-axis - its image is point A'; and reflected in y-axis - its image is point A''.



Worked example 3.5.4.

Find the image of the triangle with vertices at $A(-3,-1), B(4,0)$ and $C(1,3)$ after reflection in the y-axis.

Reflecting in the y-axis corresponds to changing the sign of the x-coordinate of every point. The image of the triangle ABC will be a triangle $A'B'C'$ with $A'=(3,-1), B'=(-4,0)$ and $C'=(-1,3)$.



Exercise 3.5.4. Find the image of triangle with vertices at $P(-4,1), Q(1,-1)$ and $R(0,3)$ after

(a) reflection in y-axis

(b) reflection in x-axis

(c) reflection in y-axis, followed by translation by $\vec{v} = \begin{pmatrix}1\\2\end{pmatrix}$

(d) translation by $\vec{v} = \begin{pmatrix}1\\2\end{pmatrix}$, followed by reflection in y-axis

(e) reflection in x-axis, followed by translation by $\vec{v} = \begin{pmatrix}-2\\1\end{pmatrix}$

(f) translation by $\vec{v} = \begin{pmatrix}-2\\1\end{pmatrix}$, followed by reflection in x-axis

Worked example 3.5.5.

The line given by the equation $y = \frac{1}{2}x + 1$ is reflected in the x-axis. Find the equation of the resulting line and sketch both lines.

We have

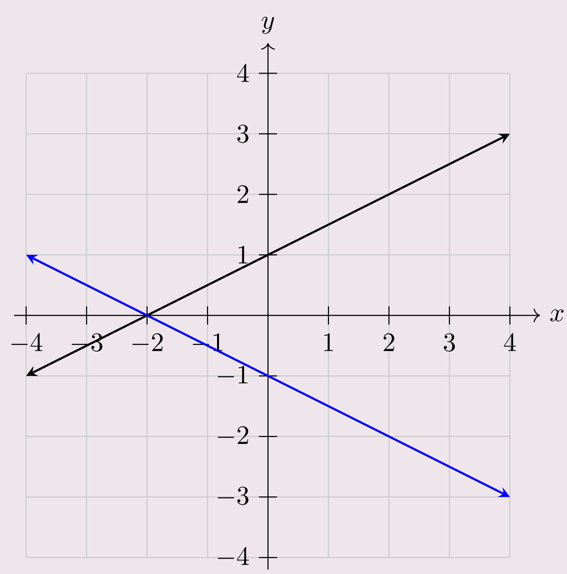
$$\begin{cases} x' = x \\ y' = -y \end{cases}$$

which gives $x = x'$ and $y = -y'$, so the equation of the image is:

$$-y' = \frac{1}{2}x' + 1$$

which gives $y' = -\frac{1}{2}x' - 1$.

The diagram below shows both lines (black before and blue after reflection).

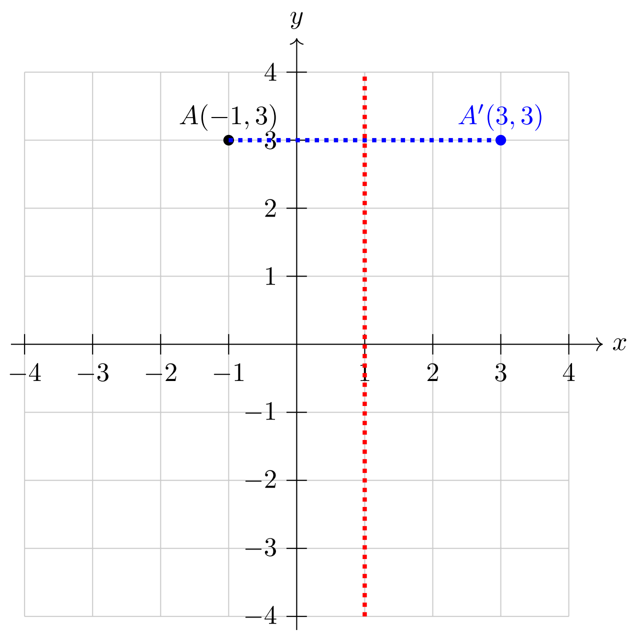


Exercise 3.5.5. Line l or curve C is reflected in the given axis. Find the equation of the resulting line/curve and sketch both lines/curves.

(a) $l : y = -\frac{1}{2}x + 2,\quad x\text{-axis}$ (b) $l : y = x - 3,\quad y\text{-axis}$ (c) $l : 2x + y = 2,\quad x\text{-axis}$

(d) $C : y = x^2,\quad x\text{-axis}$ (e) $C : y = x^2,\quad y\text{-axis}$ (f) $C : y = \sqrt{x},\quad y\text{-axis}$

When reflecting in a vertical line which is not the y-axis, the y-coordinate is unchanged while the x-coordinate changes as follows. Consider point $A(-1, 3)$ and the line $x = 1$. The image of A after reflection in $x = 1$ will be the point $A'(3, 3)$ as illustrated below.



Note that since y is unchanged we have $y' = y$. Now let's consider two cases. If x is on the left hand side of the vertical line $x = a$ (as in the example above), then the distance from A to the line is $a - x$ and the new coordinate is $x' = a + (a - x) = 2a - x$. If x is on the right hand side, then the distance is $x - a$ and the new coordinate is $x' = a - (x - a) = 2a - x$. In both cases we have $x' = 2a - x$. Of course, if A lies on the line $x = a$, then the x-coordinate remains unchanged (note that $x' = 2a - x$ still works in this case). This shows that the image of $A(x, y)$ after reflection in $x = a$ is the point $A'(x', y')$ with

$$\begin{cases} x' = 2a - x \\ y' = y \end{cases}$$

Using the same argument it can be shown that the image of $A(x, y)$ after reflection in $y = b$ is the point $A'(x', y')$ with

$$\begin{cases} x' = x \\ y' = 2b - y' \end{cases}$$

Worked example 3.5.6.

The curve given by the equation $y = \sqrt{x}$ is reflected in the line $x = 2$. Find the equation of the resulting curves and sketch both curves.

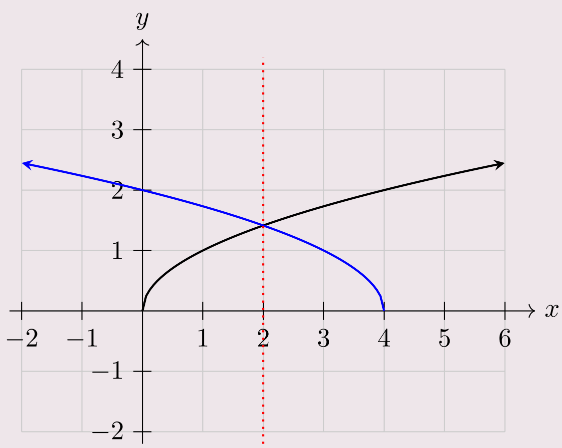
We have

$$\begin{cases} x' = 4 - x \\ y' = y \end{cases}$$

which gives $x = 4 - x'$ and $y = y'$, so the equation of the image is:

$$y' = \sqrt{4 - x'}$$

The diagram below shows both curves (black before and blue after reflection).



Exercise 3.5.6a
Find the image of the quadrilateral with vertices $A(-2, -3), B(1, -1), C(1, 3)$ and $D(-3, 3)$ after reflection in:

(a) $x = -1$ (b) $y = 1$ (c) $x = 2$

Exercise 3.5.6b
Line l is reflected in the given line k. Find the equation of the resulting line and sketch both lines.

(a) $l : y = 4 - x, \quad k : x = 1$ (b) $l : y = x - 1, \quad k : y = -2$ (c) $l : 3x + 2y = 3, \quad k : y = 1$

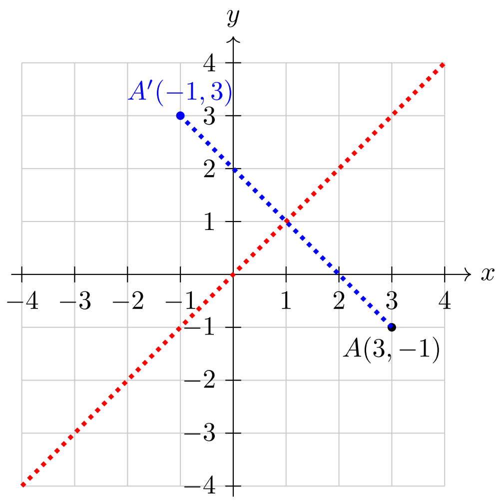
Exercise 3.5.6c
Curve C is reflected in the given line k. Find the equation of the resulting curve and sketch both curves.

(a) $C : y = x^2, \quad k : y = 1$ (b) $C : y = x^2, \quad k : x = 1$ (c) $C : y = \sqrt{x}, \quad k : x = 1$

When reflecting in a line $y = x$, the x and y coordinates of every point are interchanged. That is we get:

$$\begin{cases} x' = y \\ y' = x \end{cases}$$

This is illustrated in the diagram below.



Worked example 3.5.7.

The curve given by the equation $y = x^2$ is reflected in the line $y = x$. Find the equation of the resulting curve and sketch both curves.

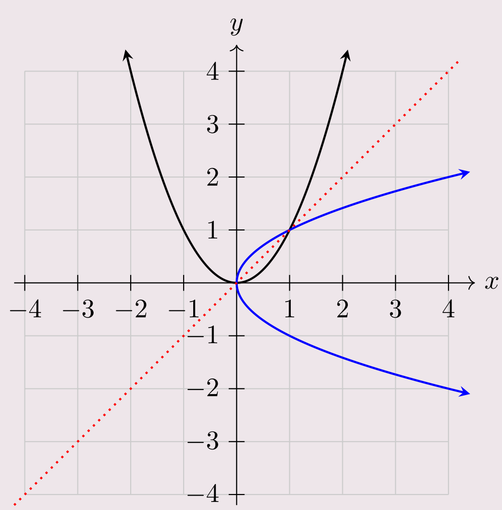
We have

$$\begin{cases} x' = y \\ y' = x \end{cases}$$

which gives $x = y'$ and $y = x'$, so the equation of the image is:

$$x' = (y')^2$$

The diagram below shows both curves (black before and blue after reflection).



Exercise 3.5.7a
Find the image of the triangle with vertices $A(-3,1), B(2,-1)$ and $C(0,4)$ after reflection in the line $y=x$.

Exercise 3.5.7b
Line l is reflected in the line $y=x$. Find the equation of the resulting line and sketch both lines.

(a) $l: y=\frac{1}{2}x+1$ (b) $l: y=2-x$ (c) $l: 2x+3y=2$

Exercise 3.5.7c
Curve C is reflected in the line $y=x$. Find the equation of the resulting curve and sketch both curves.

(a) $C: y=2^x$ (b) $C: xy=2$ (c) $C: y=\sqrt{x}$

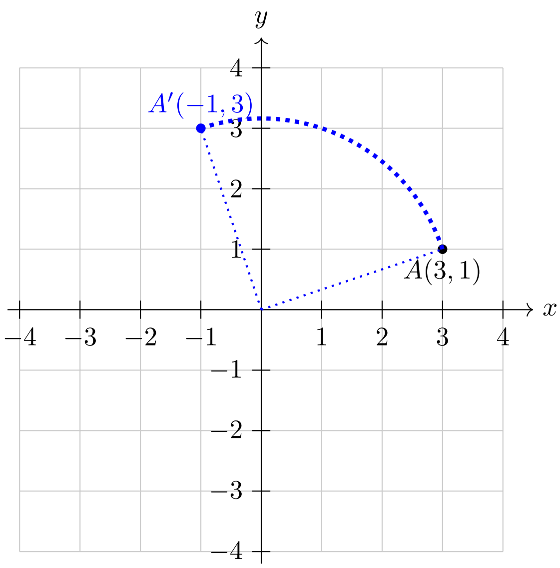
Rotating any point around the origin by 90° **counterclockwise** results in the following change in the coordinates:

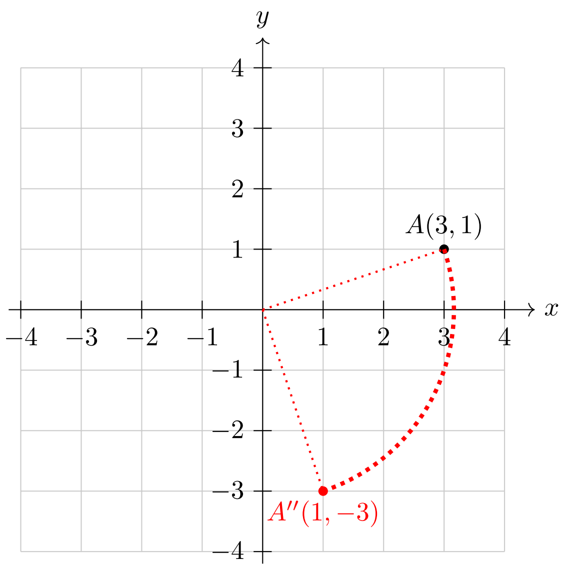
$$\begin{cases} x' = -y \\ y' = x \end{cases}$$

Similarly rotating around the origin by 90° **clockwise** results in:

$$\begin{cases} x' = y \\ y' = -x \end{cases}$$

This is illustrated in the diagram below, where point $A(3,1)$ is rotated counterclockwise (blue) and clockwise (red) 90° around the origin to points A' and A'' respectively.





Worked example 3.5.8.

The curve given by the equation $y = 2^x$ is rotated by 90° clockwise around the origin. Find the equation of the resulting curve and sketch both curves.

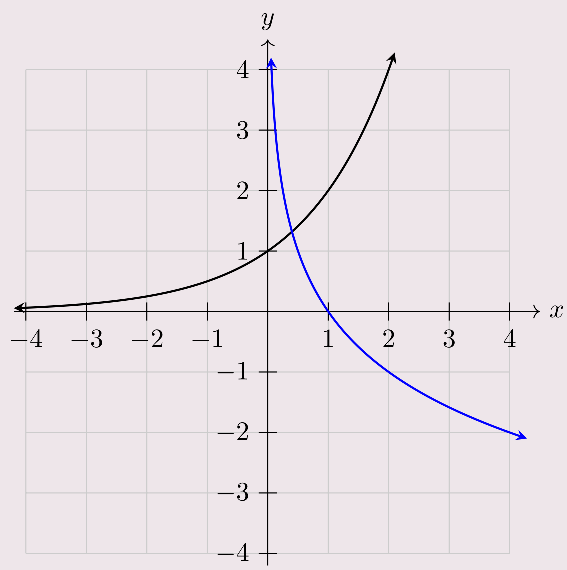
We have

$$\begin{cases} x' = y \\ y' = -x \end{cases}$$

which gives $x = -y'$ and $y = x'$, so the equation of the image is:

$$x' = 2^{-y'}$$

The diagram below shows both curves (black before and blue after reflection).



Exercise 3.5.8a
Find the image of the quadrilateral with vertices $A(-1,-2), B(3,2), C(4,2)$ and $D(0,1)$ after rotation around the origin by 90°

(a) counterclockwise

(b) clockwise

Exercise 3.5.8b
Find the equation and sketch the line $y = 1 - \frac{1}{2}x$ after rotation around the origin by 90°

(a) counterclockwise

(b) clockwise

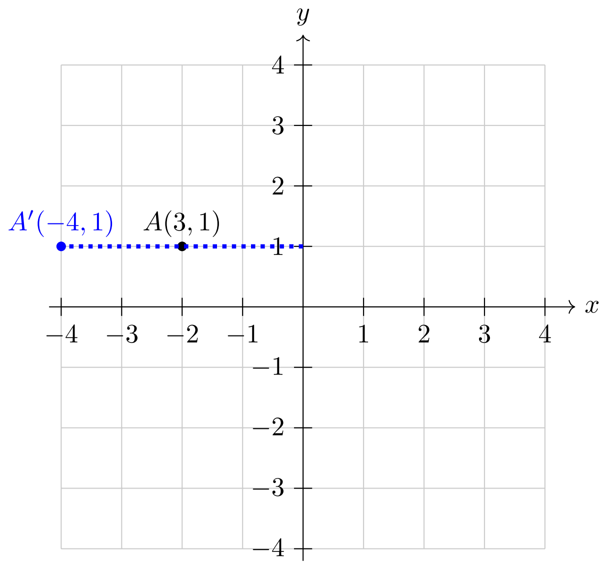
Exercise 3.5.8c
Find the equation and sketch the curve $y = \sqrt{x}$ after rotation around the origin by 90°

(a) counterclockwise

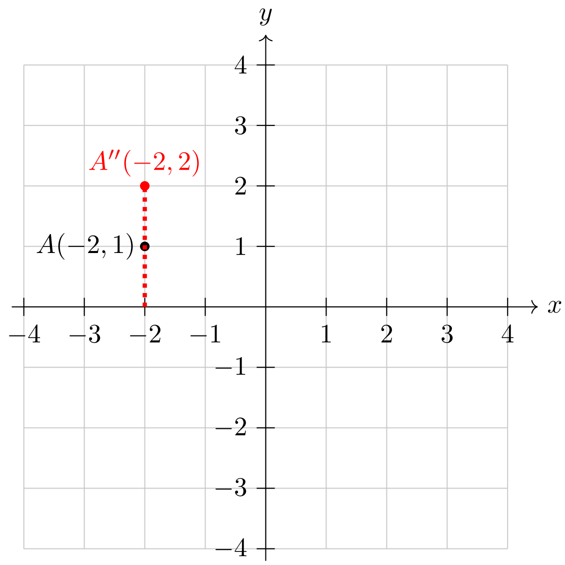
(b) clockwise

When the plane is stretched (dilated) horizontally (parallel to x-axis) with the y-axis fixed, then the distance from every point to the y-axis increase by a factor of the stretch. That is if the stretch is of factor 2, the distance increase 2-fold. Similarly, if the plane is stretched (dilated) vertically (parallel to y-axis) with the x-axis fixed, then the distance from every point to the x-axis increase by a factor of the stretch. This is illustrated in the diagram below with point $A(-2,1)$ and its images A' (after horizontal stretch by a factor of 2) and A'' (after vertical stretch by a factor of 2).

$$\begin{cases} x' = kx \\ y' = y \end{cases}$$



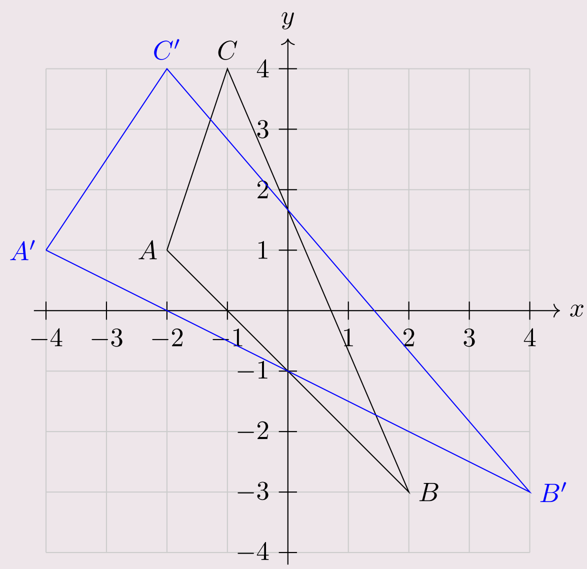
$$\begin{cases} x' = x \\ y' = ky \end{cases}$$



Worked example 3.5.9.

Find the image of the triangle with vertices at $A(-2,1), B(2,-3)$ and $C(-1,4)$ after dilation by a factor of 2 parallel to the x-axis with y-axis fixed (horizontal stretch).

The distance from every point to the y-axis doubles, so the new vertices are $A'(-4,1), B'(4,-3)$ and $C'(-2,4)$



Worked example 3.5.10.

The curve given by the equation $y = \sqrt{x} - 1$ is dilated by a factor of 2 parallel to y-axis with x-axis fixed. Find the equation of the resulting curve and sketch both curves.

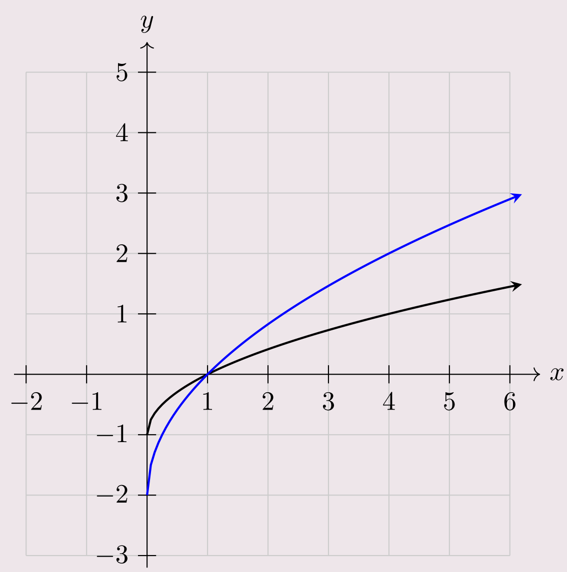
We have

$$\begin{cases} x' = x \\ y' = 2y \end{cases}$$

which gives $x = x'$ and $y = \frac{1}{2}y'$, so the equation of the image is:

$$\frac{1}{2}y' = \sqrt{x'} - 1$$

This can be written as $y' = 2\sqrt{x'} - 2$. The diagram below shows both curves (black before and blue after dilation).



Exercise 3.5.10a
Find the image of the triangle with vertices $A(-2,-2), B(2,-1)$ and $C(1,3)$ dilation by a factor of k parallel to x-axis with y-axis fixed for:

(a) $k = 2$ (b) $k = \frac{1}{2}$

Exercise 3.5.10b
Find the image of the triangle with vertices $A(-3,-1), B(4,-3)$ and $C(0,1)$ dilation by a factor of k parallel to y-axis with x-axis fixed for:

(a) $k = 3$ (b) $k = \frac{1}{1}$

Exercise 3.5.8c
Find the equation and sketch the line $y = \frac{1}{2}x - 2$ after a strech by a factor of 2

(a) parallel to x-axis (y-axis fixed) (b) parallel to y-axis (x-axis fixed)

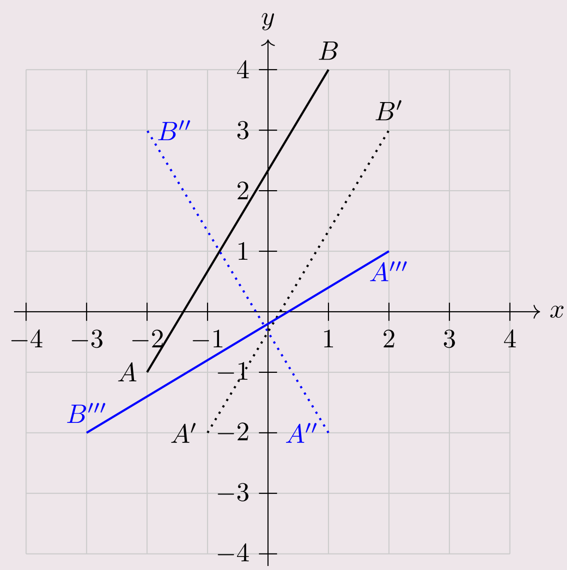
Worked example 3.5.11.

Find the image of the line segment with endpoints $A(-2,-1)$ and $B(1,4)$ after translation by $\vec{v} = \begin{pmatrix} 1 \\ -1 \end{pmatrix}$, followed by reflection in y-axis, followed by counter-clockwise rotation by $90°$ around the origin.

We will do the transformations one step at a time:

$$A(-2,-1) \xrightarrow{T_{\binom{1}{-1}}} A'(-1,-2) \xrightarrow{Ref_{x=0}} A''(1,-2) \xrightarrow{Rot_{90°}} A'''(2,1)$$

$$B(1,4) \xrightarrow{T_{\binom{1}{-1}}} B'(2,3) \xrightarrow{Ref_{x=0}} B''(-2,3) \xrightarrow{Rot_{90°}} B'''(-3,-2)$$



Worked example 3.5.12.

The curve given by the equation $y = x^2$ is translated by $\vec{v} = \begin{pmatrix} 2 \\ 1 \end{pmatrix}$, then reflected in the line $y = 2$ and rotated clockwise around the origin by $90°$. Find the equation of the resulting curve and sketch both curves.

We have:

$$\begin{cases} x' = x + 2 \\ y' = y + 1 \end{cases}$$

$$\begin{cases} x'' = x' \\ y'' = 4 - y' \end{cases}$$

$$\begin{cases} x''' = y'' \\ y''' = -x'' \end{cases}$$

which gives:

$$\begin{cases} x''' = 3 - y \\ y''' = -x - 2 \end{cases}$$

So $y = 3 - x'''$ and $x = -y''' - 2$ and the equation becomes $3 - x''' = (-y''' - 2)^2$ or $x''' = 3 - (y''' + 2)^2$